

# G-TYPE SPACES OF ULTRADISTRIBUTIONS OVER $\mathbb{R}_+^d$ AND THE WEYL PSEUDO-DIFFERENTIAL OPERATORS WITH RADIAL SYMBOLS

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**ABSTRACT.** The first part of the paper is devoted to the  $G$ -type spaces i.e. the spaces  $G_\alpha^\alpha(\mathbb{R}_+^d)$ ,  $\alpha \geq 1$  and their duals which can be described as analogous to the Gelfand-Shilov spaces and their duals but with completely new justification of obtained results. The Laguerre type expansions of the elements in  $G_\alpha^\alpha(\mathbb{R}_+^d)$ ,  $\alpha \geq 1$  and their duals characterise these spaces through the exponential and sub-exponential growth of coefficients. We provide the full topological description and by the nuclearity of  $G_\alpha^\alpha(\mathbb{R}_+^d)$ ,  $\alpha \geq 1$  the kernel theorem is proved. The second part is devoted to the class of the Weyl operators with radial symbols belonging to the  $G$ -type spaces. The continuity properties of this class of pseudo-differential operators over the Gelfand-Shilov type spaces and their duals are proved. In this way the class of the Weyl pseudo-differential operators is extended to the one with the radial symbols with the exponential and sub-exponential growth rate.

## 1. INTRODUCTION

The aim of this paper is twofold. In the first part, we study the spaces  $G_\alpha^\alpha(\mathbb{R}_+^d)$ ,  $\alpha \geq 1$  and their strong duals, analogous to the Gelfand-Shilov spaces and their duals, while in the second part, we study a class of the Weyl operators with rotationally invariant symbols using the results from the first part.

The Gelfand-Shilov spaces  $S_\alpha(\mathbb{R}^d)$ ,  $S^\beta(\mathbb{R}^d)$  and  $S_\alpha^\beta(\mathbb{R}^d)$ ,  $\alpha + \beta \geq 1$  (referred to as  $S$ -type spaces) are very well known and used in the analysis of Cauchy problems, spectral analysis, time-frequency analysis and many other fields of mathematics (see [3], [10]-[13], [19]). The spaces  $S_\alpha^\alpha(\mathbb{R}^d)$ ,  $\alpha \geq 1/2$  are of the special interest because they are invariant under Fourier transform and have been characterised through the Hermite expansions and the corresponding estimates of coefficients (see [2], [18], [21]).

On the other hand, the corresponding function spaces defined on  $\mathbb{R}_+^d$  are less studied, although they should have similar importance. Such spaces,  $G_\alpha(\mathbb{R}_+^d)$ ,  $G^\beta(\mathbb{R}_+^d)$  and  $G_\alpha^\beta(\mathbb{R}_+^d)$ ,  $\alpha + \beta \geq 2$  (referred to as  $G$ -type spaces) in the one dimensional case  $d = 1$ , were introduced by A. Duran in [7] in order to extend the Hankel-Clifford transform, analogous to the Fourier transform for the positive real line, to a class of functionals larger than that of tempered distributions on the positive real line.

In the first part, we are interested in the spaces  $G_\alpha^\alpha(\mathbb{R}_+^d)$ ,  $\alpha \geq 1$ , invariant under the Hankel-Clifford transform. In the case  $d = 1$  these spaces have been characterised through the Laguerre expansions and the corresponding estimates of the coefficients in [8]. Our investigations are connected with the papers of A. Duran [4]-[8] in the case  $d = 1$ . These papers contain a lot of fine results but, however, there exist

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subtle gaps which we improved upon. Let us briefly present our investigations. In Section 3, following [7], we introduce the spaces  $G_\alpha^\beta(\mathbb{R}_+^d)$ ,  $\alpha, \beta \geq 0$ . In Section 4, we improve upon the gaps in [6] concerning the Hankel-Clifford transform as a continuous mapping from  $\mathcal{S}(\mathbb{R}_+)$  into the same space, defining this transform for the  $d$ -dimensional case. Moreover, we introduce, as an important novelty of the paper, the modified fractional powers of the partial Hankel-Clifford transform as a main tool for an examination of the  $G$ -type spaces. In Section 5, the expansion of elements from  $G_\alpha^\alpha(\mathbb{R}_+^d)$ ,  $\alpha \geq 1$ , with respect to the Laguerre orthonormal basis, is presented. We characterise these spaces through the coefficient estimates. The main corrections of gaps in [8], related to the analytic function  $F(w)$ ,  $w \in \mathbf{D}$ , as well as the equivalence of the conditions of Proposition 5.4, are done. We underline, as an important novelty of our paper, the topological structure described in Section 6, since the explanation of this structure in the case  $d = 1$  given in [8] is inadequate. This is essentially improved in the multi-dimensional case (Theorems 6.1, 6.2) by the closed graph De Wilde theorem. Moreover, as a main consequence of the analysed topological structure, we prove in Section 6 the nuclearity of the spaces  $G_\alpha^\alpha(\mathbb{R}_+^d)$ ,  $\alpha \geq 1$  as well as the Schwartz's kernel theorem,

$$G_\alpha^\alpha(\mathbb{R}_+^{d_1}) \hat{\otimes} G_\alpha^\alpha(\mathbb{R}_+^{d_2}) \cong G_\alpha^\alpha(\mathbb{R}_+^{d_1+d_2}), \alpha \geq 1.$$

In the second part, we use the obtained series expansions in order to introduce a new class of pseudo-differential operators with radial symbols and prove continuity properties of such operators on the Gelfand- Shilov spaces and their duals. More precisely, we prove the continuity of the Weyl pseudo-differential operators with radial symbols from the spaces  $G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d)$  and  $(G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$ ,  $\alpha \geq 1/2$ . In the first case, we show that the class of the Weyl pseudo-differential operators with radial symbols is a continuous and linear mapping from  $S_\alpha^\alpha(\mathbb{R}^d)$  into  $S_\alpha^\alpha(\mathbb{R}^d)$  which can be extended to a continuous and linear mapping from  $(S_\alpha^\alpha(\mathbb{R}^d))'$  into  $S_\alpha^\alpha(\mathbb{R}^d)$ , while in the second case of the symbol, we show that the class of the Weyl pseudo-differential operator is a continuous and linear mapping from  $S_\alpha^\alpha(\mathbb{R}^d)$  into  $S_\alpha^\alpha(\mathbb{R}^d)$  which can be extended to a continuous and linear mapping from  $(S_\alpha^\alpha(\mathbb{R}^d))'$  into  $(S_\alpha^\alpha(\mathbb{R}^d))'$ . This second case is especially important since we have symbols in the dual spaces as well as the corresponding mapping over the duals of the Gelfand-Shilov spaces.

As a consequence, in Section 7, we give the corresponding results related to the symbols in  $\mathcal{S}(\mathbb{R}_+^d)$  and its dual and the corresponding continuous linear mappings related to the Schwartz space  $\mathcal{S}(\mathbb{R}^d)$  and its dual. With these special cases, we extend the corresponding results of M. W. Wong [27, Chapter 24].

## 2. PRELIMINARIES

We denote by  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  the sets of positive integers, integers, real and complex numbers, respectively;  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ,  $\mathbb{R}_+ = (0, \infty)$ . The symbol  $\mathbb{R}_+^d$  stands for  $(0, \infty)^d$  and  $\overline{\mathbb{R}_+^d}$  for its closure, i.e.  $[0, \infty)^d$ . We use the standard multi-index notation. We denote by  $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^d$ . Thus, for  $z \in \mathbb{C}^d$ ,  $z^{\mathbf{1}}$  stands for  $z_1 \cdot \dots \cdot z_d$ . Moreover, for  $x, t \in \overline{\mathbb{R}_+^d}$ ,  $x^t = x_1^{t_1} \cdot \dots \cdot x_d^{t_d}$ ; in this case we use the convention  $0^0 = 1$ . For  $n \in \mathbb{N}_0^d$ ,  $D^n$  stands for  $\partial^n / \partial t_1^{n_1} \dots \partial t_d^{n_d}$ . We use  $\|\cdot\|_2$  for the norm in the Banach space (abbreviated as a  $(B)$ -space)  $L^2(\mathbb{R}_+^d)$ .

We denote by  $\mathfrak{s}$  the space of all complex sequences  $\{a_n\}_{n \in \mathbb{N}_0^d}$  such that for each

$j \in \mathbb{N}$ ,  $\sup_{n \in \mathbb{N}_0^d} |a_n|(|n| + 1)^j < \infty$ . With these seminorms,  $\mathfrak{s}$  becomes a nuclear Fréchet space; from now on abbreviated as an  $(FN)$ -space (see [26, p. 527]; clearly in the definition of  $\mathfrak{s}$ , we can take the  $l^p$ -norms,  $p \geq 1$  instead of the sup-norm). Its strong dual, which we denote by  $\mathfrak{s}'$ , consists of all complex valued sequences  $\{a_n\}_{n \in \mathbb{N}_0^d}$  such that  $\sup_{n \in \mathbb{N}_0^d} |a_n|(|n| + 1)^{-j} < \infty$  for some  $j \in \mathbb{N}$  (which depends on  $\{a_n\}_{n \in \mathbb{N}_0^d}$ ). It is a  $(DFN)$ -space.

Let  $\alpha \geq 1$  and  $a > 1$ . We define  $\mathfrak{s}^{\alpha,a}$  to be the space of all complex sequences  $\{a_n\}_{n \in \mathbb{N}_0^d}$  for which  $\|\{a_n\}_{n \in \mathbb{N}_0^d}\|_{\mathfrak{s}^{\alpha,a}} = \sup_{n \in \mathbb{N}_0^d} |a_n| a^{|n|^{1/\alpha}} < \infty$ . With this norm  $\mathfrak{s}^{\alpha,a}$  becomes a  $(B)$ -space. For  $a > b > 1$ ,  $\mathfrak{s}^{\alpha,a}$  is continuously injected into  $\mathfrak{s}^{\alpha,b}$ . As a locally convex space (abbreviated as an l.c.s.) we define  $\mathfrak{s}^\alpha = \varinjlim_{a \rightarrow 1^+} \mathfrak{s}^{\alpha,a}$ ; the inductive limit is indeed a (Hausdorff) l.c.s. since  $\mathfrak{s}^{\alpha,a}$  are continuously injected into  $\mathfrak{s}$ .

**Proposition 2.1.** *For  $a > b > 1$ , the canonical inclusion  $\mathfrak{s}^{\alpha,a} \rightarrow \mathfrak{s}^{\alpha,b}$  is nuclear. In particular,  $\mathfrak{s}^\alpha$  is a nuclear  $(DFS)$ -space (i.e. a  $(DFN)$ -space) and its strong dual  $(\mathfrak{s}^\alpha)'$  is an  $(FN)$ -space.*

*Proof.* Since the canonical inclusion  $\mathfrak{s}^{\alpha,a} \rightarrow \mathfrak{s}^{\alpha,b}$  is a composition of two inclusions of the same type it is enough to prove that it is quasi-nuclear (for the definition of a quasi-nuclear mapping see [20, Definition 3.2.3., p. 56] and for the fact that the composition of two quasi-nuclear mappings is nuclear see [20, Theorem 3.3.2., p. 62]). For each  $m \in \mathbb{N}_0^d$ , we define  $e_m \in (\mathfrak{s}^{\alpha,a})'$  by  $\langle e_m, \{a_n\}_{n \in \mathbb{N}_0^d} \rangle = a_m b^{|m|^{1/\alpha}}$ . One easily verifies that  $\|e_m\|_{(\mathfrak{s}^{\alpha,a})'} \leq (b/a)^{|m|^{1/\alpha}}$ , hence  $\sum_{m \in \mathbb{N}_0^d} \|e_m\|_{(\mathfrak{s}^{\alpha,a})'} < \infty$ . For  $\{a_n\}_{n \in \mathbb{N}_0^d} \in \mathfrak{s}^{\alpha,a}$  we have

$$\|\{a_n\}_{n \in \mathbb{N}_0^d}\|_{\mathfrak{s}^{\alpha,b}} \leq \sum_{m \in \mathbb{N}_0^d} |a_m| b^{|m|^{1/\alpha}} = \sum_{m \in \mathbb{N}_0^d} |\langle e_m, \{a_n\}_{n \in \mathbb{N}_0^d} \rangle|,$$

i.e. the canonical inclusion  $\mathfrak{s}^{\alpha,a} \rightarrow \mathfrak{s}^{\alpha,b}$  is quasi nuclear. □ □

For the moment, denote by  $\tilde{\mathfrak{s}}^\alpha$  the space of all complex valued sequences  $\{b_n\}_{n \in \mathbb{N}_0^d}$  such that for each  $a > 1$ ,  $\|\{b_n\}_{n \in \mathbb{N}_0^d}\|_{\tilde{\mathfrak{s}}^\alpha,a} = \sum_{n \in \mathbb{N}_0^d} |b_n| a^{-|n|^{1/\alpha}} < \infty$ . With these seminorms  $\tilde{\mathfrak{s}}^\alpha$  becomes an  $(F)$ -space. Denote by  $\Xi$  the mapping  $\tilde{\mathfrak{s}}^\alpha \rightarrow (\mathfrak{s}^\alpha)'$ ,  $\langle \Xi(\{b_n\}_n), \{a_n\}_n \rangle = \sum_n a_n b_n$ . One easily verifies that it is a well defined bijection. Let  $B \subseteq \tilde{\mathfrak{s}}^\alpha$  be bounded. If  $B_1 \subseteq \mathfrak{s}^\alpha$  is bounded, there exists  $a > 1$  such that  $B_1 \subseteq \mathfrak{s}^{\alpha,a}$  and it is bounded there ( $\mathfrak{s}^\alpha$  is a  $(DFN)$ -space). Now one easily verifies that  $\sup_{\{b_n\}_n \in B, \{a_n\}_n \in B_1} |\langle \Xi(\{b_n\}_n), \{a_n\}_n \rangle| < \infty$ , i.e.  $\Xi$  maps bounded sets into bounded. Since  $\tilde{\mathfrak{s}}^\alpha$  and  $(\mathfrak{s}^\alpha)'$  are an  $(F)$ -spaces,  $\Xi$  is continuous and now the open mapping theorem verifies that  $\Xi$  is an isomorphism. Hence, we proved the following result.

**Proposition 2.2.** *The strong dual  $(\mathfrak{s}^\alpha)'$  of  $\mathfrak{s}^\alpha$  is an  $(FN)$ -space of all complex valued sequences  $\{b_n\}_{n \in \mathbb{N}_0^d}$  such that, for each  $a > 1$ ,  $\|\{b_n\}_{n \in \mathbb{N}_0^d}\|_{(\mathfrak{s}^\alpha)',a} = \sum_{n \in \mathbb{N}_0^d} |b_n| a^{-|n|^{1/\alpha}} < \infty$ . Its topology is generated by the system of seminorms  $\|\cdot\|_{(\mathfrak{s}^\alpha)',a}$ .*

Let  $\alpha \geq 1/2$ . For  $A > 0$ , denote by  $\mathcal{S}_{\alpha,A}^{\alpha,A}(\mathbb{R}^d)$  a  $(B)$ -space of all  $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$  with the norm  $\sup_{n,m \in \mathbb{N}_0^d} \|x^m D^n \varphi(x)\|_{L^2(\mathbb{R}^d)} / (A^{|n|+|m|} n!^\alpha m!^\alpha) < \infty$ . The Gelfand-Shilov space  $\mathcal{S}_\alpha(\mathbb{R}^d)$  is defined to be the space  $\varinjlim_{A \rightarrow \infty} \mathcal{S}_{\alpha,A}^{\alpha,A}(\mathbb{R}^d)$ . One easily verifies that for  $A_1 < A_2$  the canonical inclusion  $\mathcal{S}_{\alpha,A_1}^{\alpha,A_1}(\mathbb{R}^d) \rightarrow \mathcal{S}_{\alpha,A_2}^{\alpha,A_2}(\mathbb{R}^d)$  is a compact mapping, i.e.  $\mathcal{S}_\alpha(\mathbb{R}^d)$  is a  $(DFS)$ -space (for the properties of  $\mathcal{S}_\alpha(\mathbb{R}^d)$  we refer to [19, Chapter 6]; see also [10], [13]).

The Hermite polynomials and the corresponding Hermite functions are given by

$$H_j(t) = (-1)^j e^{t^2} \frac{d^j}{dt^j} (e^{-t^2}), \quad t \in \mathbb{R}, \quad h_j(t) = (2^j j! \sqrt{\pi})^{-1/2} e^{-t^2/2} H_j(t), \quad t \in \mathbb{R}, \quad j \in \mathbb{N}_0.$$

For  $n \in \mathbb{N}_0^d$ , put  $h_n(x) = h_{n_1}(x_1) \cdots h_{n_d}(x_d)$ . Then  $\{h_n\}_{n \in \mathbb{N}_0^d}$  is an orthonormal basis for  $L^2(\mathbb{R}^d)$ . For each  $n \in \mathbb{N}_0^d$ ,  $h_n \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$ . Moreover, for  $\alpha \geq 1/2$ ,  $\mathcal{S}_\alpha(\mathbb{R}^d)$  is given through the Hermite expansions. In fact, we have the following result for which the proof is similar to the proof of [16, Theorem 3.4 and Corollary 3.5] and we omit it.

**Proposition 2.3.** *Let  $\alpha \geq 1/2$ . The mapping  $\mathcal{S}_\alpha(\mathbb{R}^d) \rightarrow \mathfrak{s}^{2\alpha}$ ,  $f \mapsto \{\langle f, h_n \rangle\}_{n \in \mathbb{N}_0^d}$ , is a topological isomorphism. For  $f \in \mathcal{S}_\alpha(\mathbb{R}^d)$ ,  $\sum_{n \in \mathbb{N}_0^d} \langle f, h_n \rangle h_n$  converges absolutely to  $f$  in  $\mathcal{S}_\alpha(\mathbb{R}^d)$ .*

*The mapping  $(\mathcal{S}_\alpha(\mathbb{R}^d))' \rightarrow (\mathfrak{s}^{2\alpha})'$ ,  $T \mapsto \{\langle T, h_n \rangle\}_{n \in \mathbb{N}_0^d}$ , is a topological isomorphism. For  $T \in (\mathcal{S}_\alpha(\mathbb{R}^d))'$ ,  $\sum_{n \in \mathbb{N}_0^d} \langle T, h_n \rangle h_n$  converges absolutely to  $T$  in  $(\mathcal{S}_\alpha(\mathbb{R}^d))'$ .*

For  $j \in \mathbb{N}_0$  and  $\gamma \geq 0$ , the  $j$ -th Laguerre polynomial of order  $\gamma$  is defined by

$$L_j^\gamma(t) = \frac{t^{-\gamma} e^t}{j!} \frac{d^j}{dt^j} (e^{-t} t^{\gamma+j}), \quad t \geq 0.$$

The  $j$ -th Laguerre function of order  $\gamma$  is defined by  $\mathcal{L}_j^\gamma(t) = (j!/\Gamma(j+\gamma+1))^{1/2} L_j^\gamma(t) e^{-t/2}$ . For  $n \in \mathbb{N}_0^d$  and  $\gamma \in \overline{\mathbb{R}_+^d}$ ,  $L_n^\gamma(t) = \prod_{l=1}^d L_{n_l}^{\gamma_l}(t_l)$  and  $\mathcal{L}_n^\gamma(t) = \prod_{l=1}^d \mathcal{L}_{n_l}^{\gamma_l}(t_l)$  are the  $d$ -dimensional Laguerre polynomials and Laguerre functions of order  $\gamma$ , respectively. In the case  $\gamma = 0$ , we write  $L_n$  and  $\mathcal{L}_n$  instead of  $L_n^0$  and  $\mathcal{L}_n^0$ , respectively.

For  $\gamma \in \overline{\mathbb{R}_+^d}$  we denote by  $L^2(\mathbb{R}_+^d, t^\gamma dt)$  a  $(B)$ -space of all measurable functions on  $\mathbb{R}_+^d$  such that  $\int_{\mathbb{R}_+^d} |f(t)|^2 t^\gamma dt < \infty$ ; its norm is defined by the square root of the last quantity. Moreover,  $\{\mathcal{L}_n^\gamma\}_{n \in \mathbb{N}_0^d}$  is an orthonormal basis for  $L^2(\mathbb{R}_+^d, t^\gamma dt)$ .

Recall, (see [14]) an  $(F)$ -space  $\mathcal{S}(\mathbb{R}_+^d)$  consists of all  $f \in \mathcal{C}^\infty(\mathbb{R}_+^d)$  such that all derivatives  $D^p f$ ,  $p \in \mathbb{N}_0^d$ , extend to continuous functions on  $\overline{\mathbb{R}_+^d}$  and  $\sup_{x \in \mathbb{R}_+^d} x^k |D^p f(x)| < \infty$ ,  $\forall k, p \in \mathbb{N}_0^d$ . We denote by  $(\mathcal{S}(\mathbb{R}_+^d))'$  its strong dual.

In [4], [28] and [22] the expansions of the functions from  $\mathcal{S}(\mathbb{R}_+^d)$  with respect to the Laguerre polynomials are studied. The  $d$ -dimensional case is considered in [14]. We state these results here and refer to [14] for their proofs.

**Theorem 2.4.** ([14, Theorem 3.1]) *For  $f \in \mathcal{S}(\mathbb{R}_+^d)$  let  $a_n(f) = \int_{\mathbb{R}_+^d} f(x) \mathcal{L}_n(x) dx$ . Then  $f = \sum_{n \in \mathbb{N}_0^d} a_n(f) \mathcal{L}_n$  and the series converges absolutely in  $\mathcal{S}(\mathbb{R}_+^d)$ . Moreover the mapping  $\iota : \mathcal{S}(\mathbb{R}_+^d) \rightarrow \mathfrak{s}$ ,  $\iota(f) = \{a_n(f)\}_{n \in \mathbb{N}_0^d}$  is a topological isomorphism.*

**Theorem 2.5.** ([14, Theorem 3.2]) *For  $T \in \mathcal{S}'(\mathbb{R}_+^d)$  let  $b_n(T) = \langle T, \mathcal{L}_n \rangle$ . Then  $T = \sum_{n \in \mathbb{N}_0^d} b_n(T) \mathcal{L}_n$  and  $\{b_n(T)\}_{n \in \mathbb{N}_0^d} \in \mathfrak{s}'$  and the series converges absolutely in  $\mathcal{S}'(\mathbb{R}_+^d)$ . Conversely, if  $\{b_n\}_{n \in \mathbb{N}_0^d} \in \mathfrak{s}'$  there exists  $T \in \mathcal{S}'(\mathbb{R}_+^d)$  such that  $T = \sum_{n \in \mathbb{N}_0^d} b_n \mathcal{L}_n$ . As a consequence,  $\mathcal{S}'(\mathbb{R}_+^d)$  is topologically isomorphic to  $\mathfrak{s}'$ .*

Note that the topological isomorphisms between  $\mathcal{S}(\mathbb{R}_+^d)$  and  $\mathfrak{s}$  and between  $(\mathcal{S}(\mathbb{R}_+^d))'$  and  $\mathfrak{s}'$  imply that  $\mathcal{S}(\mathbb{R}_+^d)$  is an  $(FN)$ -space and  $(\mathcal{S}(\mathbb{R}_+^d))'$  is a  $(DFN)$ -space. In particular, the  $\pi$  and the  $\epsilon$  topologies coincide on  $\mathcal{S}(\mathbb{R}_+^{d_1}) \otimes \mathcal{S}(\mathbb{R}_+^{d_2})$  and on  $(\mathcal{S}(\mathbb{R}_+^{d_1}))' \otimes (\mathcal{S}(\mathbb{R}_+^{d_2}))'$ .

**Theorem 2.6.** ([14, Theorem 4.2]) *The following canonical isomorphisms hold:*

$$\mathcal{S}(\mathbb{R}_+^{d_1}) \hat{\otimes} \mathcal{S}(\mathbb{R}_+^{d_2}) \cong \mathcal{S}(\mathbb{R}_+^{d_1+d_2}), \quad (\mathcal{S}(\mathbb{R}_+^{d_1}))' \hat{\otimes} (\mathcal{S}(\mathbb{R}_+^{d_2}))' \cong (\mathcal{S}(\mathbb{R}_+^{d_1+d_2}))'.$$

**Theorem 2.7.** ([14, Theorem 4.3]) *The restriction mapping  $f \mapsto f|_{\mathbb{R}_+^d}$ ,  $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}_+^d)$  is a topological homomorphism onto.*

*The space  $\mathcal{S}(\mathbb{R}_+^d)$  is topologically isomorphic to the quotient space  $\mathcal{S}(\mathbb{R}^d)/N$ , where  $N = \{f \in \mathcal{S}(\mathbb{R}^d) \mid \text{supp } f \subseteq \mathbb{R}^d \setminus \mathbb{R}_+^d\}$ . Consequently,  $(\mathcal{S}(\mathbb{R}_+^d))'$  can be identified with the closed subspace of  $(\mathcal{S}(\mathbb{R}^d))'$  which consists of all tempered distributions with support in  $\overline{\mathbb{R}_+^d}$ .*

**Remark 2.8.** *The fact that  $(\mathcal{S}(\mathbb{R}_+^d))'$  is canonically isomorphic to the closed subspace of  $(\mathcal{S}(\mathbb{R}^d))'$  which consists of all tempered distributions with support in  $\overline{\mathbb{R}_+^d}$  allows us to define unambiguously the notion of derivatives of the elements of  $(\mathcal{S}(\mathbb{R}_+^d))'$ . In fact, for  $T \in (\mathcal{S}(\mathbb{R}_+^d))'$  and  $n \in \mathbb{N}_0^d$ ,  $D^n T$  stands for the  $D^n$ -derivative of  $T$  in  $(\mathcal{S}(\mathbb{R}^d))'$  sense. Since  $\text{supp } D^n T \subseteq \overline{\mathbb{R}_+^d}$ ,  $D^n T$  is a well defined element of  $(\mathcal{S}(\mathbb{R}_+^d))'$ . Moreover, by  $\mathcal{S}(\mathbb{R}_+^d) \cong \mathcal{S}(\mathbb{R}^d)/N$  (see Theorem 2.7)*

$$\langle D^n T, \varphi \rangle = (-1)^{|n|} \langle T, D^n \varphi \rangle, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}_+^d).$$

*It is important to stress that if  $T$  is given by  $\psi \in \mathcal{S}(\mathbb{R}_+^d)$  then  $D^n T$  does not have to coincide with the classical  $D^n$ -derivative of  $\psi$  (unless  $\psi$  can be extended to a smooth function on  $\mathbb{R}^d$  with support in  $\overline{\mathbb{R}_+^d}$ ). Considering  $\psi$  as an element of  $(\mathcal{S}(\mathbb{R}_+^d))'$  automatically means extending it by 0 on  $\mathbb{R}^d \setminus \overline{\mathbb{R}_+^d}$ . Of course, this extension does not have to be smooth.*

### 3. DEFINITION AND BASIC PROPERTIES OF THE SPACES $G_\alpha^\beta(\mathbb{R}_+^d)$ , $\alpha, \beta \geq 0$

Unless otherwise stated,  $\alpha$  and  $\beta$  are two positive reals. In the sequel, we will often tacitly apply the following inequalities (we use  $0^0 = 1$ )

$$m^{\alpha m} n^{\alpha n} \leq (m+n)^{\alpha(m+n)}, \quad (m+n)^{\alpha(m+n)} \leq e^{\alpha|m+n|} m^{\alpha m} n^{\alpha n}, \quad \forall m, n \in \mathbb{N}_0^d, \quad \forall \alpha \geq 0.$$

We define the basic test spaces (cf. [7, Definition 2.1] for  $d=1$ ):

Let  $A > 0$ . We denote by  $G_{\alpha, A}^{\beta, A}(\mathbb{R}_+^d)$  the space of all  $f \in \mathcal{S}(\mathbb{R}_+^d)$  for which

$$\sup_{p, k \in \mathbb{N}_0^d} \frac{\|t^{(p+k)/2} D^p f(t)\|_2}{A^{|p+k|} k^{(\alpha/2)k} p^{(\beta/2)p}} < \infty.$$

With the following seminorms

$$\sigma_{A,j}(f) = \sup_{p,k \in \mathbb{N}_0^d} \frac{\|t^{(p+k)/2} D^p f(t)\|_{L^2(\mathbb{R}_+^d)}}{A^{|p+k|} k^{(\alpha/2)k} p^{(\beta/2)p}} + \sup_{\substack{|p| \leq j \\ |k| \leq j}} \sup_{t \in \mathbb{R}_+^d} |t^k D^p f(t)|, \quad j \in \mathbb{N}_0,$$

one easily verifies that it becomes an  $(F)$ -space. Clearly, if  $A_1 < A_2$ ,  $G_{\alpha,A_1}^{\beta,A_1}(\mathbb{R}_+^d)$  is continuously injected into  $G_{\alpha,A_2}^{\beta,A_2}(\mathbb{R}_+^d)$ . Define  $G_\alpha^\beta(\mathbb{R}_+^d) = \varinjlim_{A \rightarrow \infty} G_{\alpha,A}^{\beta,A}(\mathbb{R}_+^d)$ . Since all the injections  $G_{\alpha,A}^{\beta,A} \rightarrow \mathcal{S}(\mathbb{R}_+^d)$  are continuous,  $G_\alpha^\beta(\mathbb{R}_+^d)$  is indeed a (Hausdorff) l.c.s.. Clearly,  $G_\alpha^\beta(\mathbb{R}_+^d)$  is continuously injected into  $\mathcal{S}(\mathbb{R}_+^d)$ . As inductive limit of an  $(F)$ -spaces,  $G_\alpha^\beta(\mathbb{R}_+^d)$  is a barrelled and bornological l.c.s..

For  $A > 0$  we define  $G_{\alpha,A}(\mathbb{R}_+^d)$  to be the space of all  $f \in \mathcal{S}(\mathbb{R}_+^d)$  such that

$$\sup_{k \in \mathbb{N}_0^d} \frac{\|t^{(p+k)/2} D^p f(t)\|_2}{A^{|k|} k^{(\alpha/2)k}} < \infty, \quad \forall p \in \mathbb{N}_0^d$$

and similarly,  $G^{\beta,A}(\mathbb{R}_+^d)$  to be the space of all  $f \in \mathcal{S}(\mathbb{R}_+^d)$  such that

$$\sup_{p \in \mathbb{N}_0^d} \frac{\|t^{(p+k)/2} D^p f(t)\|_2}{A^{|p|} p^{(\beta/2)p}} < \infty, \quad \forall k \in \mathbb{N}_0^d.$$

If we equip  $G_{\alpha,A}(\mathbb{R}_+^d)$  with the system of seminorms

$$\sigma'_{A,j}(f) = \sup_{\substack{|p| \leq j \\ k \in \mathbb{N}_0^d}} \sup \frac{\|t^{(p+k)/2} D^p f(t)\|_{L^2(\mathbb{R}_+^d)}}{A^{|k|} k^{(\alpha/2)k}} + \sup_{\substack{|p| \leq j \\ |k| \leq j}} \sup_{t \in \mathbb{R}_+^d} |t^k D^p f(t)|, \quad j \in \mathbb{N}_0,$$

one easily verifies that it becomes an  $(F)$ -space. Analogously, by equipping  $G^{\beta,A}(\mathbb{R}_+^d)$  with the system of seminorms

$$\sigma''_{A,j}(f) = \sup_{\substack{|k| \leq j \\ p \in \mathbb{N}_0^d}} \sup \frac{\|t^{(p+k)/2} D^p f(t)\|_{L^2(\mathbb{R}_+^d)}}{A^{|p|} p^{(\beta/2)p}} + \sup_{\substack{|p| \leq j \\ |k| \leq j}} \sup_{t \in \mathbb{R}_+^d} |t^k D^p f(t)|, \quad j \in \mathbb{N}_0,$$

it is also an  $(F)$ -space. Similarly as above, we define  $G_\alpha(\mathbb{R}_+^d) = \varinjlim_{A \rightarrow \infty} G_{\alpha,A}(\mathbb{R}_+^d)$  and  $G^\beta(\mathbb{R}_+^d) = \varinjlim_{A \rightarrow \infty} G^{\beta,A}(\mathbb{R}_+^d)$ . Thus,  $G_\alpha(\mathbb{R}_+^d)$  and  $G^\beta(\mathbb{R}_+^d)$  are barrelled and bornological l.c.s. that are continuously injected into  $\mathcal{S}(\mathbb{R}_+^d)$ .

For each  $m \in \mathbb{N}_0^d$ ,  $f(t) \mapsto t^m f(t)$  is a continuous mapping  $G_\alpha(\mathbb{R}_+^d) \rightarrow G_\alpha(\mathbb{R}_+^d)$ ,  $G^\beta(\mathbb{R}_+^d) \rightarrow G^\beta(\mathbb{R}_+^d)$  and  $G_\alpha^\beta(\mathbb{R}_+^d) \rightarrow G_\alpha^\beta(\mathbb{R}_+^d)$ .

We denote by  $(G^\beta(\mathbb{R}_+^d))'$ ,  $(G_\alpha(\mathbb{R}_+^d))'$  and  $(G_\alpha^\beta(\mathbb{R}_+^d))'$  the strong duals of  $G^\beta(\mathbb{R}_+^d)$ ,  $G_\alpha(\mathbb{R}_+^d)$  and  $G_\alpha^\beta(\mathbb{R}_+^d)$ , respectively.

One easily verifies that when  $\alpha, \beta \geq 1$ ,  $\mathcal{L}_n \in G_\alpha^\beta(\mathbb{R}_+^d)$  and hence  $G_\alpha^\beta(\mathbb{R}_+^d)$  is dense in  $\mathcal{S}(\mathbb{R}_+^d)$ . In particular, for  $\alpha \geq 1$ ,  $G_\alpha(\mathbb{R}_+^d)$ ,  $G^\alpha(\mathbb{R}_+^d)$  and  $G_\alpha^\alpha(\mathbb{R}_+^d)$  are dense in  $\mathcal{S}(\mathbb{R}_+^d)$ . Hence,  $(\mathcal{S}(\mathbb{R}_+^d))'$  is continuously injected into  $(G_\alpha(\mathbb{R}_+^d))'$ ,  $(G^\alpha(\mathbb{R}_+^d))'$  and  $(G_\alpha^\alpha(\mathbb{R}_+^d))'$ .

**Remark 3.1.** Let  $\alpha, \beta > 0$ . Then the spaces  $G_\alpha^\beta(\mathbb{R}_+^d)$  are non-trivial when  $\alpha + \beta \geq 2$ . We refer to [7, Corollary 3.9] for  $d=1$ . For  $d$ -dimensional case it follows considering

the function  $\varphi(t) = \varphi_1(t_1) \dots \varphi_d(t_d)$ , where  $\varphi_j$ ,  $j = 1, \dots, d$ , is a non-zero element of  $G_\alpha^\beta(\mathbb{R}_+)$ .

#### 4. THE HANKEL-CLIFFORD TRANSFORM

Let  $\mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+^d})$  be a  $(B)$ -space of all continuous functions  $f : \overline{\mathbb{R}_+^d} \rightarrow \mathbb{C}$  such that  $\sup_{x \in \overline{\mathbb{R}_+^d}} |f(x)| < \infty$ ; the norm of  $f \in \mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+^d})$  is given by the left-hand side.

For  $\gamma \geq 0$ , we denote by  $J_\gamma$  and  $I_\gamma$  the Bessel function of the first kind and the modified Bessel function of the first kind, respectively. Denote  $\mathbf{T}^{(d)} = \{z \in \mathbb{C}^d \mid |z_l| = 1, z_l \neq 1, \forall l = 1, \dots, d\}$ . For  $z \in \mathbf{T}^{(d)}$  and  $\gamma \in \overline{\mathbb{R}_+^d}$ , we define the fractional powers and the modified fractional powers of the Hankel-Clifford transform of  $f \in \mathcal{S}(\mathbb{R}_+^d)$  by

$$\begin{aligned} \mathcal{I}_{z,\gamma} f(t) &= \left( \prod_{l=1}^d (1 - z_l)^{-1} e^{-\frac{1}{2} \frac{1+z_l}{1-z_l} t_l} \right) \int_{\mathbb{R}_+^d} f(x) \prod_{l=1}^d e^{-\frac{1}{2} \frac{1+z_l}{1-z_l} x_l} (x_l t_l z_l)^{-\gamma_l/2} x_l^{\gamma_l} I_{\gamma_l} \left( \frac{2\sqrt{x_l t_l z_l}}{1 - z_l} \right) dx \\ \mathcal{J}_{z,\gamma} f(t) &= \left( \prod_{l=1}^d (1 - z_l)^{-1} \right) \int_{\mathbb{R}_+^d} f(x) \prod_{l=1}^d (x_l t_l z_l)^{-\gamma_l/2} x_l^{\gamma_l} I_{\gamma_l} \left( \frac{2\sqrt{x_l t_l z_l}}{1 - z_l} \right) dx. \end{aligned}$$

Since  $z \in \mathbf{T}^{(d)}$ ,  $z_l = e^{i\theta_l}$  where  $\theta_l \in (-\pi, \pi] \setminus \{0\}$ ,  $l = 1, \dots, d$ . Observe that  $(1 + z_l)/(1 - z_l)$  is purely imaginary. Moreover,  $2\sqrt{x_l t_l z_l}/(1 - z_l) = i\sqrt{x_l t_l}/\sin(\theta_l/2)$  and  $(x_l t_l z_l)^{-\gamma_l/2} = (x_l t_l)^{-\gamma_l/2} e^{-i\theta_l \gamma_l/2}$ . Hence, for  $l = 1, \dots, d$ ,

$$((x_l t_l z_l)^{-\gamma_l/2} I_{\gamma_l} \left( \frac{2\sqrt{x_l t_l z_l}}{1 - z_l} \right)) = e^{-i\theta_l \gamma_l/2} (x_l t_l)^{-\gamma_l/2} e^{(i\gamma_l \pi \operatorname{sgn} \theta_l)/2} J_{\gamma_l} \left( \frac{\sqrt{x_l t_l}}{|\sin(\theta_l/2)|} \right).$$

By the definition of the Bessel function of the first kind, it is clear that for  $\nu \geq 0$ ,  $\xi^{-\nu} |J_\nu(\xi)|$  is uniformly bounded when  $\xi \in (0, c)$  for arbitrary but fixed  $c \geq 1$ . Combining this with [1, 9.2.1, p. 364], we obtain that there exists  $C \geq 1$  such that

$$(2) \quad \left| \prod_{l=1}^d (x_l t_l z_l)^{-\gamma_l/2} I_{\gamma_l} \left( \frac{2\sqrt{x_l t_l z_l}}{1 - z_l} \right) \right| \leq C, \quad \forall x, t \in \mathbb{R}_+^d.$$

Moreover, for  $\nu \geq 0$ , by the definition of  $J_\nu$ , the function  $\xi \mapsto \xi^{-\nu} J_\nu(\xi)$ ,  $\mathbb{R}_+ \rightarrow \mathbb{C}$ , can be extended to a continuous function on  $\overline{\mathbb{R}_+}$ . Hence, (1) and (2) imply that for  $f \in \mathcal{S}(\mathbb{R}_+^d)$  the integrals in the definition for  $\mathcal{I}_{z,\gamma} f$  and  $\mathcal{J}_{z,\gamma} f$  converge absolutely i.e.  $\mathcal{I}_{z,\gamma} f, \mathcal{J}_{z,\gamma} f \in \mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+^d})$ . when  $f_j \rightarrow f$  in  $\mathcal{S}(\mathbb{R}_+^d)$ ,  $\mathcal{I}_{z,\gamma} f_j \rightarrow \mathcal{I}_{z,\gamma} f$  and  $\mathcal{J}_{z,\gamma} f_j \rightarrow \mathcal{J}_{z,\gamma} f$  in  $\mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+^d})$ . Hence,  $\mathcal{I}_{z,\gamma}$  and  $\mathcal{J}_{z,\gamma}$  are well defined continuous mappings from  $\mathcal{S}(\mathbb{R}_+^d)$  to  $\mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+^d})$ . Our goal is to prove that  $\mathcal{I}_{z,\gamma}$  and  $\mathcal{J}_{z,\gamma}$  are continuous mappings from  $\mathcal{S}(\mathbb{R}_+^d)$  to  $\mathcal{S}(\mathbb{R}_+^d)$ . Firstly, we prove this for  $\mathcal{J}_{z,\gamma}$  in the case  $d = 1$ .

**Lemma 4.1.** *For  $z \in \mathbf{T}^{(1)}$  and  $\gamma \geq 0$ ,  $\mathcal{J}_{z,\gamma}$  is a continuous mapping from  $\mathcal{S}(\mathbb{R}_+)$  into  $\mathcal{S}(\mathbb{R}_+)$ .*

*Proof.* Clearly  $\mathcal{S}(\mathbb{R}_+)$  is continuously injected into  $L^2(\mathbb{R}_+, t^\gamma dt)$ . Let  $E_\gamma$  be the operator

$$E_\gamma = tD^2 + D - \frac{t}{4} - \frac{\gamma^2}{4t} + \frac{\gamma+1}{2} = D(tD) - \frac{t}{4} - \frac{\gamma^2}{4t} + \frac{\gamma+1}{2}.$$

Then  $E_\gamma(t^{\gamma/2}\mathcal{L}_n^\gamma(t)) = -nt^{\gamma/2}\mathcal{L}_n^\gamma(t)$  (see [9, (11), p. 188]). For  $f \in \mathcal{S}(\mathbb{R}_+)$  we have

$$E_\gamma(t^{\gamma/2}f(t)) = t^{\gamma/2}((\gamma+1)Df(t) + tD^2f(t) - tf(t)/4 + (\gamma+1)f(t)/2).$$

Hence, for  $k \in \mathbb{N}$ ,  $E_\gamma^k(t^{\gamma/2}f(t)) = t^{\gamma/2}g_k(t)$  for some  $g_k \in \mathcal{S}(\mathbb{R}_+)$ . Let  $a_n(f) = \int_0^\infty f(t)\mathcal{L}_n^\gamma(t)t^\gamma dt$ . Then, by integration by parts, we have

$$\int_0^\infty g_1(t)\mathcal{L}_n^\gamma(t)t^\gamma dt = \int_0^\infty E_\gamma(t^{\gamma/2}f(t))\mathcal{L}_n^\gamma(t)t^{\gamma/2}dt = -na_n(f).$$

Iterating this, we obtain

$$\int_0^\infty g_k(t)\mathcal{L}_n^\gamma(t)t^\gamma dt = (-n)^k a_n(f).$$

Since  $g_k \in \mathcal{S}(\mathbb{R}_+) \subseteq L^2(\mathbb{R}_+, t^\gamma dt)$ , we conclude  $\{a_n(f)\}_{n \in \mathbb{N}_0} \in \mathfrak{s}$ . Observe that  $f = \sum_n a_n(f)\mathcal{L}_n^\gamma$  in  $L^2(\mathbb{R}_+, t^\gamma dt)$ . We need the following estimate for the derivatives of the Laguerre polynomials (see [5, Theorem 1]):

$$\left| t^k D^p(e^{-t/2}L_n^\gamma(t)) \right| \leq 2^{-\min\{\gamma, k\}} 4^k (n+1) \cdots (n+k) \binom{n + \max\{\gamma - k, 0\} + p}{n},$$

for all  $t \geq 0$ ,  $n, k, p \in \mathbb{N}_0$ . Denote by  $[\gamma]$  the integral part of  $\gamma$ , we have

$$\binom{n + \max\{\gamma - k, 0\} + p}{n} \leq \binom{n + [\gamma] + 1 + p}{n} \leq (n + [\gamma] + p + 1)^{[\gamma] + p + 1}.$$

Hence, there exists  $C_{p,k} \geq 1$  which depends on  $p$  and  $k$ , but not on  $n$ , such that

$$(3) \quad \left| t^k D^p \mathcal{L}_n^\gamma(t) \right| \leq C_{p,k} (n+1)^{k+p+[\gamma]+1}.$$

Since  $\{a_n(f)\}_n \in \mathfrak{s}$ , we have  $\sum_n |a_n(f)| \sup_{t \in \mathbb{R}_+} |t^k D^p \mathcal{L}_n^\gamma(t)| < \infty$ , i.e.  $\sum_n a_n(f)\mathcal{L}_n^\gamma$  converges absolutely in  $\mathcal{S}(\mathbb{R}_+)$ . Since  $\mathcal{J}_{z,\gamma}f : \mathcal{S}(\mathbb{R}_+) \rightarrow \mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+})$  is continuous,  $\mathcal{J}_{z,\gamma}f = \sum_n a_n(f)\mathcal{J}_{z,\gamma}\mathcal{L}_n^\gamma$  and the series converges absolutely in  $\mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+})$ . We need the following equality (see [17, (4.20.3), p. 83])

$$(4) \quad \int_0^\infty J_\gamma(\sqrt{xt})x^{\gamma/2}\mathcal{L}_n^\gamma(x)dx = 2(-1)^n t^{\gamma/2}\mathcal{L}_n^\gamma(t), \quad \gamma \geq 0, n \in \mathbb{N}_0.$$

Using (1) and (4) we obtain

$$(\mathfrak{J}_{k,\gamma}\mathcal{L}_n^\gamma(t) = 2(-1)^n e^{-i\gamma\theta/2} e^{(i\gamma\pi \operatorname{sgn} \theta)/2} (1 - e^{i\theta})^{-1} |\sin(\theta/2)|^{-\gamma} \mathcal{L}_n^\gamma(t/\sin^2(\theta/2)).$$

The estimate (3) together with (5) implies that  $\sum_n a_n(f)\mathcal{J}_{z,\gamma}\mathcal{L}_n^\gamma$  converges absolutely in  $\mathcal{S}(\mathbb{R}_+)$ . Thus, we obtain that the image of  $\mathcal{S}(\mathbb{R}_+)$  under  $\mathcal{J}_{z,\gamma}$  is contained in  $\mathcal{S}(\mathbb{R}_+)$ . Since  $\mathcal{J}_{z,\gamma} : \mathcal{S}(\mathbb{R}_+) \rightarrow \mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+})$  is continuous its graph is closed in  $\mathcal{S}(\mathbb{R}_+) \times \mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+})$ . As  $\mathcal{S}(\mathbb{R}_+)$  is continuously injected into  $\mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+})$  and  $\mathcal{J}_{z,\gamma}(\mathcal{S}(\mathbb{R}_+)) \subseteq \mathcal{S}(\mathbb{R}_+)$ , the graph of  $\mathcal{J}_{z,\gamma}$  is closed in  $\mathcal{S}(\mathbb{R}_+) \times \mathcal{S}(\mathbb{R}_+)$ . Since  $\mathcal{S}(\mathbb{R}_+)$  is an  $(F)$ -space, the closed graph theorem implies that  $\mathcal{J}_{z,\gamma} : \mathcal{S}(\mathbb{R}_+) \rightarrow \mathcal{S}(\mathbb{R}_+)$  is continuous.  $\square$   $\square$

Now, by the principle of induction, we show that for  $z \in \mathbf{T}^{(d)}$  and  $\gamma \in \overline{\mathbb{R}_+^d}$ ,  $\mathcal{J}_{z,\gamma}$  is a continuous mapping from  $\mathcal{S}(\mathbb{R}_+^d)$  into itself. When  $f \in \mathcal{S}(\mathbb{R}_+^d)$ , we denote  $\mathcal{J}_{z,\gamma}$  by  $\mathcal{J}_{z,\gamma}^{(d)}$  in order to avoid confusions. We already considered the case  $d = 1$ ;  $\mathcal{J}_{z,\gamma}^{(1)} : \mathcal{S}(\mathbb{R}_+) \rightarrow \mathcal{S}(\mathbb{R}_+)$  is continuous. Let  $\mathcal{J}_{z,\gamma}^{(d)}$  be continuous. Let  $\nu = (\gamma, \gamma') \in \overline{\mathbb{R}_+^{d+1}}$  where  $\gamma \in \overline{\mathbb{R}_+^d}$  and  $\gamma' \geq 0$  and let  $\zeta = (z, z') \in \mathbf{T}^{(d+1)}$  where  $z \in \mathbf{T}^{(d)}$  and  $z' \in \mathbf{T}^{(1)}$ . The mapping  $\mathcal{J}_{z,\gamma}^{(d)} \otimes \mathcal{J}_{z',\gamma'}^{(1)} : \mathcal{S}(\mathbb{R}_+^d) \otimes_\pi \mathcal{S}(\mathbb{R}_+) \rightarrow \mathcal{S}(\mathbb{R}_+^d) \otimes_\pi \mathcal{S}(\mathbb{R}_+)$  is continuous.



Denoting by  $\tilde{\mathcal{J}}_{\zeta,\nu}$  its continuous extension on the completions, Theorem 2.6 yields that  $\tilde{\mathcal{J}}_{\zeta,\nu}$  is a continuous mapping from  $\mathcal{S}(\mathbb{R}_+^{d+1})$  into itself. Observe that for each  $f \in \mathcal{S}(\mathbb{R}_+^d) \otimes \mathcal{S}(\mathbb{R}_+)$ ,  $\mathcal{J}_{\zeta,\nu}^{(d+1)} f(t) = \tilde{\mathcal{J}}_{\zeta,\nu} f(t)$ ,  $\forall t \in \mathbb{R}^{d+1}$ . Thus  $\mathcal{J}_{\zeta,\nu}^{(d+1)} f \in \mathcal{S}(\mathbb{R}_+^{d+1})$ . If  $f \in \mathcal{S}(\mathbb{R}_+^{d+1})$ , there exists a sequence  $f_j \in \mathcal{S}(\mathbb{R}_+^d) \otimes \mathcal{S}(\mathbb{R}_+)$ ,  $j \in \mathbb{N}$ , such that  $f_j \rightarrow f$  in  $\mathcal{S}(\mathbb{R}_+^{d+1})$  (cf. Theorem 2.6;  $\mathcal{S}(\mathbb{R}_+^{d+1})$  is an  $(F)$ -space). Since we proved that  $\mathcal{J}_{\zeta,\nu}^{(d+1)} : \mathcal{S}(\mathbb{R}_+^{d+1}) \rightarrow \mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+^{d+1}})$  is continuous (see the discussion before Lemma 4.1, we have, for each fixed  $t \in \mathbb{R}_+^{d+1}$ ,

$$\mathcal{J}_{\zeta,\nu}^{(d+1)} f(t) = \lim_{j \rightarrow \infty} \mathcal{J}_{\zeta,\nu}^{(d+1)} f_j(t) = \lim_{j \rightarrow \infty} \tilde{\mathcal{J}}_{\zeta,\nu} f_j(t) = \tilde{\mathcal{J}}_{\zeta,\nu} f(t).$$

Hence  $\mathcal{J}_{\zeta,\nu}^{(d+1)} f \in \mathcal{S}(\mathbb{R}_+^{d+1})$  and  $\mathcal{J}_{\zeta,\nu}^{(d+1)} f = \tilde{\mathcal{J}}_{\zeta,\nu} f$ ,  $\forall f \in \mathcal{S}(\mathbb{R}_+^{d+1})$ . We conclude that  $\mathcal{J}_{\zeta,\nu}^{(d+1)} : \mathcal{S}(\mathbb{R}_+^{d+1}) \rightarrow \mathcal{S}(\mathbb{R}_+^{d+1})$  is continuous.

Next we prove that  $\mathcal{J}_{z,\gamma}$  extends to isometry from  $L^2(\mathbb{R}_+^d, t^\gamma dt)$  onto itself. Firstly, we prove the following claim:

For  $\gamma \in \overline{\mathbb{R}_+^d}$ , let  $V_\gamma^{(d)} \subseteq \mathcal{S}(\mathbb{R}_+^d)$  be the space which consists of all finite linear combinations of the form  $\sum_{k \leq n} a_k \mathcal{L}_k^\gamma$ , where  $a_k \in \mathbb{C}$ . Then, for each  $\gamma \in \overline{\mathbb{R}_+^d}$ ,  $V_\gamma^{(d)}$  is dense in  $\mathcal{S}(\mathbb{R}_+^d)$ .

The proof follows by the principle of induction on the dimension. For  $d = 1$ , it is already proved in the first part of the proof of Lemma 4.1. Assume that the assertion holds for  $d \in \mathbb{N}$ . Let  $\nu = (\gamma, \gamma') \in \overline{\mathbb{R}_+^{d+1}}$  where  $\gamma \in \overline{\mathbb{R}_+^d}$  and  $\gamma' \geq 0$ . The inductive hypothesis implies that  $V_\gamma^{(d)} \otimes V_{\gamma'}^{(1)}$  is dense in  $\mathcal{S}(\mathbb{R}_+^d) \otimes_\epsilon \mathcal{S}(\mathbb{R}_+)$  and consequently in  $\mathcal{S}(\mathbb{R}_+^{d+1})$  by Theorem 2.6. One easily verifies that  $V_\gamma^{(d)} \otimes V_{\gamma'}^{(1)} \subseteq V_\nu^{(d+1)}$  and the proof is completed.

By (1) and (4), we obtain

$$\mathcal{J}_{z,\gamma} \mathcal{L}_n^\gamma(t) = 2^d (-1)^{|n|} c_{z,\gamma} \left( \prod_{l=1}^d |\sin(\theta_l/2)|^{-\gamma_l} \right) \mathcal{L}_n^\gamma \left( \frac{t_1}{\sin^2(\theta_1/2)}, \dots, \frac{t_d}{\sin^2(\theta_d/2)} \right),$$

where  $c_{z,\gamma} = \prod_{l=1}^d e^{-i\gamma_l \theta_l/2} e^{(i\gamma_l \pi \operatorname{sgn} \theta_l)/2} (1 - e^{i\theta_l})^{-1}$ . One easily verifies that the set  $\{\mathcal{J}_{z,\gamma} \mathcal{L}_n^\gamma | n \in \mathbb{N}_0^d\}$  is orthonormal in  $L^2(\mathbb{R}_+^d, t^\gamma dt)$ . Now, for  $f \in V_\gamma^{(d)}$  ( $V_\gamma^{(d)}$  is a subspace of  $\mathcal{S}(\mathbb{R}_+^d)$  defined in the assertion above) we have  $\|\mathcal{J}_{z,\gamma} f\|_{L^2(\mathbb{R}_+^d, t^\gamma dt)} = \|f\|_{L^2(\mathbb{R}_+^d, t^\gamma dt)}$ . Since  $V_\gamma^{(d)}$  is dense in  $\mathcal{S}(\mathbb{R}_+^d)$  we have  $\|\mathcal{J}_{z,\gamma} f\|_{L^2(\mathbb{R}_+^d, t^\gamma dt)} = \|f\|_{L^2(\mathbb{R}_+^d, t^\gamma dt)}$  for all  $f \in \mathcal{S}(\mathbb{R}_+^d)$ . Thus  $\mathcal{J}_{z,\gamma}$  extends to an isometry from  $L^2(\mathbb{R}_+^d, t^\gamma dt)$  into itself. Secondly, we prove the surjectivity of  $\mathcal{J}_{z,\gamma}$ . Note that  $\mathcal{J}_{\bar{z},\gamma} \mathcal{J}_{z,\gamma} \mathcal{L}_n^\gamma = \mathcal{L}_n^\gamma$  and  $\mathcal{J}_{z,\gamma} \mathcal{J}_{\bar{z},\gamma} \mathcal{L}_n^\gamma = \mathcal{L}_n^\gamma$ , where  $\bar{z} = (\bar{z}_1, \dots, \bar{z}_d)$  (it follows from (4) and (6)). Hence,  $\mathcal{J}_{z,\gamma} : L^2(\mathbb{R}_+^d, t^\gamma dt) \rightarrow L^2(\mathbb{R}_+^d, t^\gamma dt)$  is bijective with an inverse  $\mathcal{J}_{\bar{z},\gamma}$ . Incidentally, we can also conclude that  $\mathcal{J}_{z,\gamma} : \mathcal{S}(\mathbb{R}_+^d) \rightarrow \mathcal{S}(\mathbb{R}_+^d)$  is a topological isomorphism (has an inverse  $\mathcal{J}_{\bar{z},\gamma}$ ).

Let  $z \in \mathbf{T}^{(d)}$  and  $\Phi_z(t) = \prod_{l=1}^d e^{-\frac{1}{2} \frac{1+z_l}{1-z_l} t_l}$ . Since  $(1+z_l)/(1-z_l)$  is purely imaginary, for all  $l = 1, \dots, d$ ,  $|\Phi_z(t)| = 1$  and one easily verifies that the mapping  $f \mapsto \Phi_z f$ , is a topological isomorphism on  $\mathcal{S}(\mathbb{R}_+^d)$  and an isometry from  $L^2(\mathbb{R}_+^d, t^\gamma dt)$  onto itself. Since  $\mathcal{I}_{z,\gamma} f = \Phi_z \mathcal{J}_{z,\gamma} (\Phi_z f)$  we can conclude that  $\mathcal{I}_{z,\gamma}$  is topological isomorphism on  $\mathcal{S}(\mathbb{R}_+^d)$  and isometry from  $L^2(\mathbb{R}_+^d, t^\gamma dt)$  onto itself; clearly, its inverse is  $\mathcal{I}_{\bar{z},\gamma}$ .

Now, by the same technique as in the proof of [7, Lemma 3.2], we have:

$$(7) \quad \left\| t^{(p+k+\gamma)/2} D^p f(t) \right\|_2 = \left( \prod_{l=1}^d |1 - z_l|^{-p_l+k_l} \right) \left\| t^{(p+k+\gamma)/2} D^k \mathcal{J}_{z,\gamma} f(t) \right\|_2, \quad f \in \mathcal{S}(\mathbb{R}_+^d),$$

for  $\gamma \in \overline{\mathbb{R}_+^d}$  and  $p, k \in \mathbb{N}_0^d$ .

Next, we summarise the properties of  $\mathcal{J}_{z,\gamma}$  and  $\mathcal{I}_{z,\gamma}$  in the following proposition:

**Proposition 4.2.** *For  $\gamma \in \overline{\mathbb{R}_+^d}$  and  $z \in \mathbf{T}^{(d)}$  the fractional powers and the modified fractional powers of the Hankel-Clifford transform  $\mathcal{I}_{z,\gamma}$  and  $\mathcal{J}_{z,\gamma}$  are topological isomorphisms on  $\mathcal{S}(\mathbb{R}_+^d)$  and they extend to isometries from  $L^2(\mathbb{R}_+^d, t^\gamma dt)$  onto itself with inverses,  $\mathcal{I}_{\bar{z},\gamma}$  and  $\mathcal{J}_{\bar{z},\gamma}$  respectively. Moreover, for all  $p, k \in \mathbb{N}_0^d$  and  $f \in \mathcal{S}(\mathbb{R}_+^d)$ , (7) is valid.*

Notice that when  $z = -1 \in \mathbf{T}^{(d)}$  then  $\mathcal{H}_\gamma = \mathcal{J}_{z,\gamma} = \mathcal{I}_{z,\gamma}$  where  $\mathcal{H}_\gamma$  is the  $d$ -dimensional Hankel-Clifford transform, defined as

$$\mathcal{H}_\gamma(f)(t) = 2^{-d} t^{-\gamma/2} \int_{\mathbb{R}_+^d} f(x) x^{\gamma/2} \prod_{l=1}^d J_{\gamma_l}(\sqrt{x_l t_l}) dx, \quad t \in \mathbb{R}_+^d.$$

By (6),  $\mathcal{L}_n^\gamma$ ,  $n \in \mathbb{N}_0^d$ , are eigenfunctions for  $\mathcal{H}_\gamma$ ; more precisely  $\mathcal{H}_\gamma \mathcal{L}_n^\gamma = (-1)^{|n|} \mathcal{L}_n^\gamma$ .

Since  $\mathcal{J}_{z,0}$  is an isomorphism on  $\mathcal{S}(\mathbb{R}_+^d)$ , by (7) we have the following result.

**Theorem 4.3.** *The modified fractional powers of the Hankel-Clifford transform  $\mathcal{J}_{z,0}$  are isomorphisms of  $G_\alpha(\mathbb{R}_+^d)$ ,  $G^\beta(\mathbb{R}_+^d)$  and  $G_\alpha^\beta(\mathbb{R}_+^d)$  onto  $G^\alpha(\mathbb{R}_+^d)$ ,  $G_\beta(\mathbb{R}_+^d)$  and  $G_\beta^\alpha(\mathbb{R}_+^d)$  respectively.*

Proposition 4.2 is also valid for the modified fractional powers of the partial Hankel-Clifford transform. To make this precise let  $d', d'' \in \mathbb{N}$ ,  $\gamma = (\gamma', \gamma'') \in \overline{\mathbb{R}_+^{d'}} \times \overline{\mathbb{R}_+^{d''}} = \overline{\mathbb{R}_+^d}$  (for brevity  $d = d' + d''$ ) and  $z' = (z_1, \dots, z_{d'}) \in \mathbf{T}^{(d')}$ . Denote by  $\mathcal{J}_{z',\gamma'}^{d'}$  the modified fractional power of the Hankel-Clifford transform on  $\mathbb{R}_+^{d'}$  and by  $\text{Id}^{d''}$  the identity operator  $\mathcal{S}(\mathbb{R}_+^{d''}) \rightarrow \mathcal{S}(\mathbb{R}_+^{d''})$ . Theorem 2.6 and Proposition 4.2 imply that  $\mathcal{J}_{z',\gamma'}^{d'} \hat{\otimes} \text{Id}^{d''}$  is a topological isomorphism on  $\mathcal{S}(\mathbb{R}_+^d)$  (it follows that  $\mathcal{J}_{z',\gamma'}^{d'} \hat{\otimes} \text{Id}^{d''}$  is an injection from [15, Theorem 5, p. 277] and a homomorphism from [15, Theorem 7, p. 189]; note  $\mathcal{S}(\mathbb{R}_+^d)$  is nuclear). We denote by  $x \in \mathbb{R}_+^d$   $x = (x', x'')$  where  $x' = (x_1, \dots, x_{d'})$  and  $x'' = (x_{d'+1}, \dots, x_d)$ . Let  $f \in \mathcal{S}(\mathbb{R}_+^d)$ . Define

$$\mathcal{J}_{z',\gamma'}^{(d')} f(t) = \left( \prod_{l=1}^{d'} (1 - z_l)^{-1} \right) \int_{\mathbb{R}_+^{d'}} f(x', t'') \prod_{l=1}^{d'} (x_l t_l z_l)^{-\gamma_l/2} x_l^{\gamma_l} I_{\gamma_l} \left( \frac{2\sqrt{x_l t_l z_l}}{1 - z_l} \right) dx'.$$

By the same technique already described for the absolute convergence of  $\mathcal{J}_{z,\gamma}$ , one proves that  $\mathcal{J}_{z',\gamma'}^{(d')} f \in \mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+^d})$ . When  $f_j \rightarrow f$  in  $\mathcal{S}(\mathbb{R}_+^d)$ ,  $\mathcal{J}_{z',\gamma'}^{(d')} f_j \rightarrow \mathcal{J}_{z',\gamma'}^{(d')} f$  in  $\mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+^d})$ . Since  $\mathcal{J}_{z',\gamma'}^{(d')} f(t) = \mathcal{J}_{z',\gamma'}^{d'} \hat{\otimes} \text{Id}^{d''} f(t)$  for  $f \in \mathcal{S}(\mathbb{R}_+^{d'}) \otimes \mathcal{S}(\mathbb{R}_+^{d''})$ , we accomplish the same for all  $f \in \mathcal{S}(\mathbb{R}_+^d)$ . Hence, the first part of the next proposition follows.

**Proposition 4.4.** *The modified fractional power of the partial Hankel-Clifford transform  $\mathcal{J}_{z',\gamma'}^{(d')}$  is a topological isomorphism on  $\mathcal{S}(\mathbb{R}_+^d)$ .*

Moreover,  $\mathcal{J}_{z',\gamma'}^{(d')}$  extends to an isometry from  $L^2(\mathbb{R}_+^d, t^\gamma dt)$  onto itself with an inverse  $\mathcal{J}_{z',\gamma'}^{(d')}$ . For all  $(p', p''), (k', k'') \in \mathbb{N}_0^{d'} \times \mathbb{N}_0^{d''} = \mathbb{N}_0^d$  and all  $f \in \mathcal{S}(\mathbb{R}_+^d)$

$$\begin{aligned} & \left\| t^{(p'+k'+\gamma')/2} t^{(p''+k'')/2} D_t^p f(t) \right\|_2 \\ &= \left( \prod_{l=1}^{d'} |1 - z_l|^{-p_l+k_l} \right) \left\| t^{(p'+k'+\gamma')/2} t^{(p''+k'')/2} D_{t'}^{k'} D_{t''}^{p''} \mathcal{J}_{z',\gamma'}^{(d')} f(t) \right\|_2. \end{aligned}$$

*Proof.* The proof that  $\mathcal{J}_{z',\gamma'}^{(d')}$  extends to an isometry from  $L^2(\mathbb{R}_+^d, t^\gamma dt)$  onto itself with an inverse  $\mathcal{J}_{z',\gamma'}^{(d')}$  is the same as for  $\mathcal{J}_{z,\gamma}$  given above. As in the proof of [7, Lemma 3.2 iii)], one obtains for  $f \in \mathcal{S}(\mathbb{R}_+^d)$

$$(8) \quad \left\| t^{(p'+k'+\gamma')/2} t^{(p''+k'')/2} D_{t'}^{p'} f(t) \right\|_2 = \left( \prod_{l=1}^{d'} |1 - z_l|^{-p_l+k_l} \right) \left\| t^{(p'+k'+\gamma')/2} t^{(p''+k'')/2} D_{t'}^{k'} \mathcal{J}_{z',\gamma'}^{(d')} f(t) \right\|_2.$$

Clearly, for  $f \in \mathcal{S}(\mathbb{R}_+^{d'}) \otimes \mathcal{S}(\mathbb{R}_+^{d''})$ ,  $D_{t''}^{p''} \mathcal{J}_{z',\gamma'}^{(d')} f = \mathcal{J}_{z',\gamma'}^{(d')} D_{t''}^{p''} f$ . Hence, the same holds for  $f \in \mathcal{S}(\mathbb{R}_+^d)$  and the equality follows from (8).  $\square$

If  $\Lambda' = \{\lambda_1', \dots, \lambda_{d'}'\} \subseteq \{1, \dots, d\}$  and  $\Lambda'' = \{\lambda_1'', \dots, \lambda_{d''}''\} = \{1, \dots, d\} \setminus \Lambda'$  one can also consider the modified fractional power of the partial Hankel-Clifford transform with respect to  $x_{\Lambda'} = (x_{\lambda_1'}, \dots, x_{\lambda_{d'}'})$  defined by (here  $x_{\Lambda''} = (x_{\lambda_1''}, \dots, x_{\lambda_{d''}''})$ ) and abusing the notation we write  $x = (x_{\Lambda'}, x_{\Lambda''})$

$$\mathcal{J}_{z',\gamma_{\Lambda'}}^{(\Lambda')} f(t) = \left( \prod_{l=1}^{d'} (1 - z_l)^{-1} \right) \int_{\mathbb{R}_+^{d'}} f(x_{\Lambda'}, t_{\Lambda''}) \prod_{l=1}^{d'} (x_{\lambda_l'} t_{\lambda_l'} z_l)^{-\gamma_{\lambda_l'}/2} x_{\lambda_l'}^{\gamma_{\lambda_l'}} I_{\gamma_{\lambda_l'}} \left( \frac{2\sqrt{x_{\lambda_l'} t_{\lambda_l'} z_l}}{1 - z_l} \right) dx_{\Lambda'}.$$

**Corollary 4.5.** *Using the same notations as above,  $\mathcal{J}_{z',\gamma_{\Lambda'}}^{(\Lambda')}$  is a topological isomorphism on  $\mathcal{S}(\mathbb{R}_+^d)$  and it extends to an isometry from  $L^2(\mathbb{R}_+^d, t^\gamma dt)$  onto itself with an inverse  $\mathcal{J}_{z',\gamma_{\Lambda'}}^{(\Lambda')}$ . For all  $f \in \mathcal{S}(\mathbb{R}_+^d)$  and all  $(p_{\Lambda'}, p_{\Lambda''}), (k_{\Lambda'}, k_{\Lambda''}) \in \mathbb{N}_0^d$*

$$(9) \quad \left\| t_{\Lambda'}^{(p_{\Lambda'}+k_{\Lambda'}+\gamma_{\Lambda'})/2} t_{\Lambda''}^{(p_{\Lambda''}+k_{\Lambda''})/2} D_t^p f(t) \right\|_2 = \left( \prod_{l=1}^{d'} |1 - z_l|^{-p_{\lambda_l'}+k_{\lambda_l'}} \right) \left\| t_{\Lambda'}^{(p_{\Lambda'}+k_{\Lambda'}+\gamma_{\Lambda'})/2} t_{\Lambda''}^{(p_{\Lambda''}+k_{\Lambda''})/2} D_{t_{\Lambda'}}^{k_{\Lambda'}} D_{t_{\Lambda''}}^{p_{\Lambda''}} \mathcal{J}_{z',\gamma_{\Lambda'}}^{(\Lambda')} f(t) \right\|_2.$$

*Proof.* Let  $\Theta : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the orthogonal transformation given by  $\Theta(x) = y$ , where  $y_{\lambda_1'} = x_1, \dots, y_{\lambda_{d'}'} = x_{d'}, y_{\lambda_1''} = x_{d'+1}, \dots, y_{\lambda_{d''}''} = x_d$ . Observe that  $\Theta$  maps  $\mathbb{R}_+^d$  and  $\overline{\mathbb{R}_+^d}$  bijectively onto themselves. Let  $\tilde{\Theta}$  be the mapping  $f \mapsto f \circ \Theta$ ,  $L^2(\mathbb{R}_+^d) \rightarrow L^2(\mathbb{R}_+^d)$ . One easily verifies that for each  $\mu \in \overline{\mathbb{R}_+^d}$  it is an isometry from  $L^2(\mathbb{R}_+^d, t^\mu dt)$  onto  $L^2(\mathbb{R}_+^d, t^{\Theta^{-1}\mu} dt)$  and a topological isomorphism on  $\mathcal{S}(\mathbb{R}_+^d)$ . Its inverse is  $\tilde{\Theta}^{-1} f = f \circ \Theta^{-1}$ . Let  $\nu' = (\gamma_{\lambda_1'}, \dots, \gamma_{\lambda_{d'}'}) \in \overline{\mathbb{R}_+^{d'}}$ . The corollary follows from Proposition 4.4 and the fact that  $\mathcal{J}_{z',\gamma_{\Lambda'}}^{(\Lambda')} f = \tilde{\Theta}^{-1} \mathcal{J}_{z',\nu'}^{(d')} \tilde{\Theta} f$ .  $\square$

**Remark 4.6.** Observe that if  $\Lambda' = \emptyset$  then  $\mathcal{J}_{z', \gamma_{\Lambda'}}^{(\Lambda')} = \text{Id}$  and when  $\Lambda' = \{1, \dots, d\}$ ,  $\mathcal{J}_{z', \gamma_{\Lambda'}}^{(\Lambda')}$  is just  $\mathcal{J}_{z, \gamma}$ .

Let  $z' = -1 \in \mathbf{T}^{(d')}$  in  $\mathcal{J}_{z', \gamma_{\Lambda'}}^{(\Lambda')}$  we obtain the partial Hankel-Clifford transform with respect to  $x_{\Lambda'} = (x_{\lambda_1}, \dots, x_{\lambda_{d'}})$  denoted by  $\mathcal{H}_{\gamma_{\Lambda'}}^{(\Lambda')}$ .

As a direct consequence of Corollary 4.5 we have the following result.

**Corollary 4.7.**  $\mathcal{J}_{z', 0}^{(\Lambda')}$  is a topological isomorphism on  $G_{\alpha}^{\alpha}(\mathbb{R}_+^d)$  with an inverse  $\mathcal{J}_{z', 0}^{(\Lambda')}$ . In particular,  $\mathcal{H}_0^{(\Lambda')}$  is a self-inverse topological isomorphism on  $G_{\alpha}^{\alpha}(\mathbb{R}_+^d)$ .

## 5. FOURIER-LAGUERRE COEFFICIENTS IN $G_{\alpha}^{\alpha}(\mathbb{R}_+^d)$ , $\alpha \geq 1$

In this section, we characterise the space  $G_{\alpha}^{\alpha}(\mathbb{R}_+^d)$ ,  $\alpha \geq 1$  in terms of the Fourier-Laguerre coefficients.

**Proposition 5.1.** ([8, Lemma 3.1], for  $d=1$ ) Let  $f \in L^2(\mathbb{R}_+^d)$  and  $a_n = \int_{\mathbb{R}_+^d} f(t) \mathcal{L}_n(t) dt$ ,  $n \in \mathbb{N}_0^d$ . If there exist constants  $c > 0$  and  $a > 1$  such that

$$(10) \quad |a_n| \leq ca^{-|n|^{1/\alpha}}, \quad n \in \mathbb{N}_0^d,$$

then  $f \in G_{\alpha}^{\alpha}(\mathbb{R}_+^d)$ ,  $\alpha \geq 1$ .

*Proof.* As  $\{a_n\}_{n \in \mathbb{N}_0^d} \in \mathfrak{s}^{\alpha} \subseteq \mathfrak{s}$ , it follows  $f \in \mathcal{S}(\mathbb{R}_+^d)$  and the series  $\sum_n a_n \mathcal{L}_n$  converges absolutely in  $\mathcal{S}(\mathbb{R}_+^d)$  to  $f$ . Since  $n_1^{1/\alpha} + \dots + n_d^{1/\alpha} \leq d|n|^{1/\alpha}$ , denoting  $\tilde{a} = a^{1/d} > 1$ , we have  $a^{-|n|^{1/\alpha}} \leq \prod_{l=1}^d \tilde{a}^{-n_l^{1/\alpha}}$ .

Using the estimates (2.5) and (2.6) given in the proof of [8, Lemma 2.1], for  $p \in \mathbb{N}_0^d$  we have

$$(11) \quad \left\| t^{p/2} \mathcal{L}_n(t) \right\|_2 \leq 2^{|p|+5d} \prod_{l=1}^d (n_l + 1) \dots \left( n_l + \left\lceil \frac{p_l}{2} \right\rceil + 2 \right),$$

$$(12) \quad \left\| t^{p/2} D^p \mathcal{L}_n(t) \right\|_2 \leq 2^{5d} \prod_{l=1}^d (n_l + 1) \dots \left( n_l + \left\lceil \frac{p_l}{2} \right\rceil + 2 \right)$$

. Let  $\Lambda = \{\lambda_1, \dots, \lambda_{d'}\} \subseteq \{1, \dots, d\}$ . Since  $\mathcal{H}_0^{(\Lambda)} \mathcal{L}_n = (-1)^{n_{\lambda_1} + \dots + n_{\lambda_{d'}}} \mathcal{L}_n$ , (11) implies

$$\begin{aligned} \left\| t^{p/2} \mathcal{H}_0^{(\Lambda)} f(t) \right\|_2 &\leq \sum_{n \in \mathbb{N}_0^d} |a_n| \left\| t^{p/2} \mathcal{L}_n(t) \right\|_2 \\ &\leq c 2^{|p|+5d} \sum_{n \in \mathbb{N}_0^d} \prod_{l=1}^d \tilde{a}^{-n_l^{1/\alpha}} (n_l + 1) \dots \left( n_l + \left\lceil \frac{p_l}{2} \right\rceil + 2 \right) \\ &\leq c 2^{|p|+5d} \prod_{l=1}^d \tilde{a}^{([p_l/2]+2)} \sum_{n \in \mathbb{N}_0^d} \prod_{l=1}^d \tilde{a}^{-(n_l + [p_l/2]+2)^{1/\alpha}} \left( n_l + \left\lceil \frac{p_l}{2} \right\rceil + 2 \right)^{[p_l/2]+2}. \end{aligned}$$

Let  $u > 0$ ,  $v > 1$ . Clearly,  $\rho_{u,v}(x) = v^{-(x+u)^{1/\alpha}}(x+u)^u$ ,  $x \in (-u, +\infty)$  attains its maximum at  $x = (\alpha u / \ln v)^\alpha - u$ . This implies that there exist  $C_1, A_1 > 0$  such that

$$(13) \quad \left\| t^{p/2} \mathcal{H}_0^{(\Lambda)} f(t) \right\|_2 \leq C_1 A_1^{|p|} p^{(\alpha/2)p}, \text{ for all } p \in \mathbb{N}_0^d, \Lambda \subseteq \{1, \dots, d\}.$$

Similarly, by using (12), there exist  $C_2, A_2 > 0$  such that

$$(14) \quad \left\| t^{p/2} D^p \mathcal{H}_0^{(\Lambda)} f(t) \right\|_2 \leq C_2 A_2^{|p|} p^{(\alpha/2)p}, \text{ for all } p \in \mathbb{N}_0^d, \Lambda \subseteq \{1, \dots, d\}.$$

Denote by  $(\cdot, \cdot)$  the inner product in  $L^2(\mathbb{R}_+^d)$ . Since  $\mathcal{H}_0^{(\Lambda)} f \in \mathcal{S}(\mathbb{R}_+^d)$ , by integration by parts one easily verifies that

$$\left( t^{(p+k)/2} D^p \mathcal{H}_0^{(\Lambda)} f(t), t^{(p+k)/2} D^p \mathcal{H}_0^{(\Lambda)} f(t) \right) = \left| \left( D^p \left( t^{p+k} D^p \mathcal{H}_0^{(\Lambda)} f(t) \right), \mathcal{H}_0^{(\Lambda)} f(t) \right) \right|.$$

Hence, for all  $k, p \in \mathbb{N}_0^d$  such that  $2k \geq p$  by (13) and (14), we obtain

$$\begin{aligned} & \left\| t^{(p+k)/2} D^p \mathcal{H}_0^{(\Lambda)} f(t) \right\|_2^2 \\ & \leq \sum_{m \leq p} \binom{p}{m} \frac{(p+k)!}{(p+k-m)!} \left| \left( t^{p+k-m} D^{2p-m} \mathcal{H}_0^{(\Lambda)} f(t), \mathcal{H}_0^{(\Lambda)} f(t) \right) \right| \\ & \leq 2^{|p|+|k|} \sum_{m \leq p} \binom{p}{m} m! \left| \left( t^{(2p-m)/2} D^{2p-m} \mathcal{H}_0^{(\Lambda)} f(t), t^{(2k-m)/2} \mathcal{H}_0^{(\Lambda)} f(t) \right) \right| \\ & \leq C' A^{|p|+|k|} \sum_{m \leq p} \binom{p}{m} m^{(\alpha/2)m} m^{(\alpha/2)m} (2p-m)^{(\alpha/2)(2p-m)} (2k-m)^{(\alpha/2)(2k-m)} \\ & \leq C' A^{|p|+|k|} 2^{|p|} (2p)^{\alpha p} (2k)^{\alpha k}, \end{aligned}$$

i.e. there exist  $C_3, A_3 > 0$  such that for all  $k, p \in \mathbb{N}_0^d$  such that  $2k \geq p$  and all  $\Lambda \subseteq \{1, \dots, d\}$

$$(15) \quad \left\| t^{(p+k)/2} D^p \mathcal{H}_0^{(\Lambda)} f(t) \right\|_2 \leq C_3 A_3^{|p+k|} p^{(\alpha/2)p} k^{(\alpha/2)k}.$$

Let now  $p, k \in \mathbb{N}_0^d$  be arbitrary but fixed. Let  $\Lambda' = \{\lambda'_1, \dots, \lambda'_{d'}\} \subseteq \{1, \dots, d\}$  be such that  $k_{\lambda'_l} < p_{\lambda'_l}/2$ ,  $l = 1, \dots, d'$  and  $\Lambda'' = \{\lambda''_1, \dots, \lambda''_{d''}\} = \{1, \dots, d\} \setminus \Lambda'$  be such that  $k_{\lambda''_l} \geq p_{\lambda''_l}/2$ ,  $l = 1, \dots, d''$ . Then (9) and (15) imply

$$\left\| t^{(p+k)/2} D_t^p f(t) \right\|_2 \leq 2^{|k|} \left\| t^{(p+k)/2} D_{t_{\Lambda'}}^{k_{\Lambda'}} D_{t_{\Lambda''}}^{p_{\Lambda''}} \mathcal{H}_0^{(\Lambda')} f(t) \right\|_2 \leq C_3 (2A_3)^{|p+k|} p^{(\alpha/2)p} k^{(\alpha/2)k},$$

i.e.  $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$ . □

Our next goal is to prove that  $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$  implies (10). We need some preparations.

Let  $\mathbf{\Pi} = \Pi_1 \times \dots \times \Pi_d$ , where  $\Pi_l = \{z_l \in \mathbb{C} \mid \operatorname{Im} z_l < 0\}$ ,  $l = 1, \dots, d$ . One easily verifies that for each  $z = x + iy \in \mathbf{\Pi}$ , the function  $t \mapsto e^{-2\pi i z t}$ ,  $\mathbb{R}_+^d \rightarrow \mathbb{C}$ , is in  $G_\alpha^\alpha(\mathbb{R}_+^d)$  (also in  $\mathcal{S}(\mathbb{R}_+^d)$ ). Also, for  $z = x + iy \in \mathbf{\Pi}$ , the functions  $t \mapsto D_{x_l} e^{-2\pi i(x+iy)t} = -2\pi i t_l e^{-2\pi i(x+iy)t}$ ,  $\mathbb{R}_+^d \rightarrow \mathbb{C}$  and  $t \mapsto D_{y_l} e^{-2\pi i(x+iy)t} = 2\pi t_l e^{-2\pi i(x+iy)t}$ ,  $\mathbb{R}_+^d \rightarrow \mathbb{C}$  are in  $G_\alpha^\alpha(\mathbb{R}_+^d)$  (also in  $\mathcal{S}(\mathbb{R}_+^d)$ ) for  $l = 1, \dots, d$ . For the moment, denote by  $e_l$ ,  $l = 1, \dots, d$ , the point in  $\mathbb{R}^d$  which all coordinates are 0 except the  $l$ -th coordinate which is equal to 1. By standard arguments, one proves that for the fixed

$x^{(0)} = (x_1^{(0)}, \dots, x_d^{(0)}) \in \mathbb{R}^d$  and  $y^{(0)} = (y_1^{(0)}, \dots, y_d^{(0)}) \in \mathbb{R}^d$  with  $y_l^{(0)} < 0$ ,  $l = 1, \dots, d$  (i.e.  $z^{(0)} = x^{(0)} + iy^{(0)} \in \Pi$ ) we have

$$\begin{aligned} \left( e^{-2\pi i(x^{(0)} + x_l e_l + iy^{(0)})t} - e^{-2\pi i(x^{(0)} + iy^{(0)})t} \right) / x_l &\rightarrow -2\pi i t_l e^{-2\pi i(x^{(0)} + iy^{(0)})t}, \text{ as } x_l \rightarrow 0 \text{ and} \\ \left( e^{-2\pi i(x^{(0)} + i(y^{(0)} + y_l e_l))t} - e^{-2\pi i(x^{(0)} + iy^{(0)})t} \right) / y_l &\rightarrow 2\pi i t_l e^{-2\pi i(x^{(0)} + iy^{(0)})t}, \text{ as } y_l \rightarrow 0 \end{aligned}$$

in  $G_{\alpha, A}^{\alpha, A}(\mathbb{R}_+^d)$  for some  $A > 0$  and consequently in  $G_\alpha^\alpha(\mathbb{R}_+^d)$  and  $\mathcal{S}(\mathbb{R}_+^d)$ . Moreover,

$$-2\pi i t_l e^{-2\pi i(x + iy)t} \rightarrow -2\pi i t_l e^{-2\pi i(x^{(0)} + iy^{(0)})t} \text{ and } 2\pi i t_l e^{-2\pi i(x + iy)t} \rightarrow 2\pi i t_l e^{-2\pi i(x^{(0)} + iy^{(0)})t}$$

as  $(x, y) \rightarrow (x^{(0)}, y^{(0)})$  in  $G_{\alpha, A}^{\alpha, A}(\mathbb{R}_+^d)$  for some  $A > 0$ . Hence, the same holds in  $G_\alpha^\alpha(\mathbb{R}_+^d)$  and  $\mathcal{S}(\mathbb{R}_+^d)$ . It follows that for each  $u \in (G_\alpha^\alpha(\mathbb{R}_+^d))'$  or  $u \in (\mathcal{S}(\mathbb{R}_+^d))'$ , the function  $z \mapsto \mathcal{F}_\Pi u(z) = \langle u(t), e^{-2\pi i z t} \rangle$ ,  $\Pi \rightarrow \mathbb{C}$ , is of the class  $\mathcal{C}^1$ ;  $D_{x_l} \mathcal{F}_\Pi u(x + iy) = \langle u(t), D_{x_l} e^{-2\pi i(x + iy)t} \rangle$  and  $D_{y_l} \mathcal{F}_\Pi u(x + iy) = \langle u(t), D_{y_l} e^{-2\pi i(x + iy)t} \rangle$ . Since the Cauchy-Riemann equations hold for  $\mathcal{F}_\Pi u$ , it is analytic on  $\Pi$ . Let  $\mathbf{D} = D_1 \times \dots \times D_d$ , where  $D_l = \{w_l \in \mathbb{C} \mid |w_l| < 1\}$ ,  $l = 1, \dots, d$ . Observe that the mapping

$$w \mapsto \Omega(w) = ((1 + w_1)/(4\pi i(1 - w_1)), \dots, (1 + w_d)/(4\pi i(1 - w_d)))$$

is a biholomorphic mapping from  $\mathbf{D}$  onto  $\Pi$  with an inverse

$$z \mapsto \Omega^{-1}(z) = ((4\pi i z_1 - 1)/(4\pi i z_1 + 1), \dots, (4\pi i z_d - 1)/(4\pi i z_d + 1)).$$

Thus, we have the following result.

**Lemma 5.2.** *For each  $u \in (G_\alpha^\alpha(\mathbb{R}_+^d))'$  or  $u \in (\mathcal{S}(\mathbb{R}_+^d))'$ , the function*

$$\mathcal{F}_\mathbf{D} u(w) = \mathcal{F}_\Pi u(\Omega(w)) = \left\langle u(t), \prod_{l=1}^d e^{-\frac{1}{2} \frac{1+w_l}{1-w_l} t_l} \right\rangle, \quad \mathbf{D} \rightarrow \mathbb{C},$$

is analytic on  $\mathbf{D}$ , i.e.  $\mathcal{F}_\mathbf{D} u \in \mathcal{O}(\mathbf{D})$ .

**Proposition 5.3.** ([6, Proposition 1.1], for  $d=1$ ) *Let  $u \in (\mathcal{S}(\mathbb{R}_+^d))'$  and  $a_n = \langle u, \mathcal{L}_n \rangle$ ,  $n \in \mathbb{N}_0^d$ . Then,*

$$(16) \quad \mathcal{F}_\mathbf{D}(u)(w) = \prod_{j=1}^d (1 - w_j) \sum_{n \in \mathbb{N}_0^d} a_n w^n, \quad w \in \mathbf{D}.$$

In particular, if  $\mathcal{F}_\mathbf{D} u = 0$  then  $u = 0$ .

*Proof.* By Theorem 2.5,  $u = \sum_{n \in \mathbb{N}_0^d} a_n \mathcal{L}_n$  and the series converges absolutely in  $(\mathcal{S}(\mathbb{R}_+^d))'$ . As  $e^{-2\pi i z t} \in \mathcal{S}(\mathbb{R}_+^d)$ ,  $z \in \Pi$ , we obtain

$$\mathcal{F}_\Pi(u)(z) = \sum_{n \in \mathbb{N}_0^d} a_n \int_{\mathbb{R}_+^d} \mathcal{L}_n(t) e^{-2\pi i z t} dt, \quad z \in \Pi.$$

Moreover, as (see [9, p. 191])

$$\int_0^\infty t^\gamma L_n^\gamma(t) e^{-st} dt = \frac{\Gamma(n+1+\gamma)(s-1)^n}{n! s^{\gamma+n+1}}, \quad \gamma > -1, \operatorname{Re} s > 0,$$

we obtain

$$\mathcal{F}_\Pi(u)(z) = \sum_{n \in \mathbb{N}_0^d} a_n \prod_{j=1}^d \frac{(\frac{1}{2} + 2\pi i z_j - 1)^{n_j}}{(\frac{1}{2} + 2\pi i z_j)^{n_j+1}}, \quad z \in \Pi.$$

By the definition of  $\mathcal{F}_{\mathbf{D}}u$ , (16) follows.  $\square$

The next two assertions are already proved in [8], Lemma 3.2 and Corollary 3.5, in the case  $d=1$ . However, there are subtle gaps which we improve upon.

**Proposition 5.4.** *Let  $\alpha \geq 1$  and  $\{a_n\}_{n \in \mathbb{N}_0^d}$  be a sequence of complex numbers such that  $a_n \rightarrow 0$  as  $|n| \rightarrow \infty$ . Then*

$$F(w) = (1 - w)^1 \sum_{n \in \mathbb{N}_0^d} a_n w^n, \quad w \in \mathbf{D},$$

*belongs to  $\mathcal{O}(\mathbf{D})$ . The following conditions are equivalent:*

(i) *There exist constants  $C, A > 0$  such that*

$$(17) \quad |D^p F(w)| \leq CA^{|p|} p^{\alpha p}, \quad p \in \mathbb{N}_0^d, w \in \mathbf{D}.$$

(ii) *There exist constants  $c > 0, a > 1$  such that  $|a_n| \leq ca^{-|n|^{1/\alpha}}$ ,  $n \in \mathbb{N}_0^d$ .*

*Proof.* Clearly  $F \in \mathcal{O}(\mathbf{D})$ . Let  $\sum_{n \in \mathbb{N}_0^d} b_n w^n$  be the power series expansion of  $F$  at 0. Then, for  $n \in \mathbb{N}_0^d$  we have

$$\begin{aligned} b_n &= \frac{D^n F(0)}{n!} = \frac{1}{n!} \sum_{\substack{k \leq n \\ k \leq \mathbf{1}}} \binom{n}{k} (-1)^{|k|} \left( \sum_{m \geq n-k} \frac{m!}{(m-n+k)!} a_m w^{m-n+k} \right) \Big|_{w=0} \\ (18) \quad &= \sum_{\substack{k \leq n \\ k \leq \mathbf{1}}} (-1)^{|k|} a_{n-k}. \end{aligned}$$

Thus, for  $n, m \in \mathbb{N}_0^d$ ,

$$(19) \quad \sum_{p \leq m} b_{n+\mathbf{1}+p} = \sum_{p \leq m} \sum_{k \leq \mathbf{1}} (-1)^{|k|} a_{n+p+\mathbf{1}-k}.$$

Firstly, assume that  $d \geq 2$ . Denote by  $Q_{n,m}$  the  $d$ -dimensional parallelepiped  $\{x \in \mathbb{R}^d \mid n_l \leq x_l \leq n_l + m_l + 1, l = 1, \dots, d\}$ . If  $q \in \mathbb{N}_0^d$  is such that  $n+q$  is in the interior of  $Q_{n,m}$  then  $a_{n+q}$  appears exactly  $2^d$  times in the sum on the right hand side of (19) such that  $2^{d-1}$  times with the "+" sign and  $2^{d-1}$  times with "-" sign. If  $n+q$  is on the  $s$ -dimensional face of  $Q_{n,m}$ ,  $1 \leq s \leq d-1$ , then  $a_{n+q}$  appears exactly  $2^s$  times half of which are with the "+" sign and the other half with the "-" sign. Thus on the right hand side of (19) everything cancels except for those terms which indexes are the vertices of  $Q_{n,m}$  and they appear only once. For  $k \in \mathbb{N}_0^d$  with  $k \leq \mathbf{1}$  denote by  $m^{(k)}$  the multi-index that satisfies  $m_l^{(k)} = 0$  if  $k_l = 0$  and  $m_l^{(k)} = m_l + 1$  if  $k_l = 1$ ,  $l = 1, \dots, d$ ; when  $k$  varies through the multi-indexes that are  $\leq \mathbf{1}$ ,  $n+m+\mathbf{1}-m^{(k)}$  varies through the vertices of  $Q_{n,m}$ . Using this notations, by the above observations, we have

$$(20) \quad \sum_{p \leq m} b_{n+\mathbf{1}+p} = \sum_{k \leq \mathbf{1}} (-1)^{|k|} a_{n+m+\mathbf{1}-m^{(k)}}, \quad \forall n, m \in \mathbb{N}_0^d.$$

Clearly, for  $d = 1$  (19) and (20) are equal.

Assume that (i) holds. Since  $D^p F(w) = \sum_{n \geq p} (n!/(n-p)!) b_n w^{n-p}$ , the hypothesis in (i) and the Cauchy formula yield  $(n!/(n-p)!) |b_n| \leq CA^{|p|} p^{\alpha p}$ , for all  $n, p \in \mathbb{N}_0^d$ ,

$n \geq p$ . As  $n!/(n-p)! \geq e^{-|p|}n^p$ , for  $n \geq p$ , we have

$$(21) \quad |b_n| \leq C \prod_{j=1}^d \inf_{p_j \leq n_j} \frac{(eA)^{p_j} p_j^{\alpha p_j}}{n_j^{p_j}}, \quad n \in \mathbb{N}_0^d.$$

Of course we can assume  $A \geq 1$ . Then, if  $p_j \geq n_j$ ,

$$(eA)^{p_j} p_j^{\alpha p_j} / n_j^{p_j} \geq (eA)^{n_j} n_j^{\alpha n_j} / n_j^{n_j},$$

and so the infimum in (21) can be taken varying on  $p_j \geq 0$ ,  $j = 1, \dots, d$ . Thus, [10, (2) and (3), p. 169-170] imply, with suitable  $c' > 0$  and  $a' > 1$ ,

$$|b_n| \leq C \prod_{j=1}^d \inf_{p_j \geq 0} \frac{(eA)^{p_j} p_j^{\alpha p_j}}{n_j^{p_j}} \leq c' a'^{-|n|^{1/\alpha}}, \quad n \in \mathbb{N}_0^d.$$

Observe that for  $p, n \in \mathbb{N}_0^d$  with  $p \geq n$ , we have  $|p|^{1/\alpha} \geq (|p-n|^{1/\alpha} + |n|^{1/\alpha})/2$ . Thus, if we put  $a = \sqrt{a'} > 1$  we have  $a'^{-|p|^{1/\alpha}} \leq a^{-|p-n|^{1/\alpha}} a^{-|n|^{1/\alpha}}$  for all  $p \geq n$ . The above estimate for  $|b_n|$  together with (20) implies that for all  $n, m \in \mathbb{N}_0^d$

$$\begin{aligned} |a_n| &\leq \sum_{p \leq m} |b_{n+\mathbf{1}+p}| + \sum_{\substack{k \leq \mathbf{1} \\ k \neq \mathbf{1}}} |a_{n+m+\mathbf{1}-m^{(k)}}| \leq c' a^{-|n|^{1/\alpha}} \sum_{p \in \mathbb{N}_0^d} a^{-|p|^{1/\alpha}} + \sum_{\substack{k \leq \mathbf{1} \\ k \neq \mathbf{1}}} |a_{n+m+\mathbf{1}-m^{(k)}}| \\ &= c a^{-|n|^{1/\alpha}} + \sum_{\substack{k \leq \mathbf{1} \\ k \neq \mathbf{1}}} |a_{n+m+\mathbf{1}-m^{(k)}}|. \end{aligned}$$

The last sum has exactly  $2^d - 1$  terms and since  $k \neq \mathbf{1}$ ,  $|n+m+\mathbf{1}-m^{(k)}| \geq |n| + \min\{m_l | l = 1, \dots, d\}$ . Let  $n \in \mathbb{N}_0^d$  be arbitrary but fixed. Since the above estimate for  $|a_n|$  holds for arbitrary  $m \in \mathbb{N}_0^d$  and since  $a_n \rightarrow 0$  as  $|n| \rightarrow \infty$  (by hypothesis), this implies  $|a_n| \leq c a^{-|n|^{1/\alpha}}$ .

Assume now that (ii) holds. Then (18) implies the existence of  $a > 1$  and  $c > 0$  such that  $|b_n| \leq c a^{-|n|^{1/\alpha}}$ ,  $\forall n \in \mathbb{N}_0^d$ . Observe that  $n_1^{1/\alpha} + \dots + n_d^{1/\alpha} \leq d|n|^{1/\alpha}$ . Hence, by putting  $a' = a^{1/d}$ , we have  $a^{-|n|^{1/\alpha}} \leq \prod_{j=1}^d a'^{-n_j^{1/\alpha}}$ . Now, for  $p \in \mathbb{N}_0^d$  and  $w \in \mathbf{D}$  we obtain

$$|D^p F(w)| \leq \sum_{n \geq p} \frac{n!}{(n-p)!} |b_n| \leq c \sum_{n \in \mathbb{N}_0^d} \prod_{j=1}^d n_j^{p_j} a'^{-n_j^{1/\alpha}}.$$

Since  $\rho(x) = x^p u^{-x^{1/\alpha}}$ ,  $x \geq 0$  ( $u > 1$ ,  $p \in \mathbb{N}_0$ ) attains its maximum at  $x = (\alpha p / \ln u)^\alpha$ , we proved (17).  $\square$

We will prove in Proposition 5.6 that for  $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$ , the analytic function  $\mathcal{F}_{\mathbf{D}}(f)$  satisfies part (i) of the previous proposition. In order to prove this we need the next result; its proof is analogous to the proof of [8, Theorem 3.3] for the one dimensional case and we omit it.

**Proposition 5.5.** *Let  $f \in G_\alpha(\mathbb{R}_+^d)$ ,  $\alpha \geq 1$ . Then there exist constants  $C, A > 0$  such that*

$$|D^p \mathcal{F}_{\mathbf{D}}(f)(w)| \leq C A^{|p|} p^{\alpha p}, \quad p \in \mathbb{N}_0^d, \quad w \in \mathbf{D}, \quad \operatorname{Re} w_l \leq 0, \quad l = 1, \dots, d.$$



**Proposition 5.6.** *Let  $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$ ,  $\alpha \geq 1$ . Then there exist constants  $C, A > 0$  such that*

$$(22) \quad |D^p \mathcal{F}_\mathbf{D}(f)(w)| \leq CA^{|p|} p^{\alpha p}, \quad p \in \mathbb{N}_0^d, \quad w \in \mathbf{D}$$

and  $\lim_{w \rightarrow \mathbf{1}} \mathcal{F}_\mathbf{D}(f)(w) = 0$ .

*Proof.* As  $f \in \mathcal{S}(\mathbb{R}_+^d)$ , Proposition 5.3 implies that  $\lim_{w \rightarrow \mathbf{1}} \mathcal{F}_\mathbf{D}(f)(w) = 0$ .

We introduce some notation to make the simpler. Let  $\Lambda' = \{\lambda'_1, \dots, \lambda'_{d'}\} \subseteq \{1, \dots, d\}$  and  $\Lambda'' = \{\lambda''_1, \dots, \lambda''_{d''}\} = \{1, \dots, d\} \setminus \Lambda'$ . For  $\zeta \in \mathbb{C}^d$  (or in  $\mathbb{R}_+^d$ , or in  $\mathbb{N}_0^d$ ) we denote  $\zeta_{\Lambda'} = (\zeta_{\lambda'_1}, \dots, \zeta_{\lambda'_{d'}})$ ,  $\zeta_{\Lambda''} = (\zeta_{\lambda''_1}, \dots, \zeta_{\lambda''_{d''}})$  and by abusing the notation we write  $\zeta = (\zeta_{\Lambda'}, \zeta_{\Lambda''})$ . Let  $\tilde{\Lambda}'$  be the biholomorphic mapping from  $\mathbb{C}^d$  onto itself defined by  $\tilde{\Lambda}' w = \zeta$  where  $\zeta_{\lambda'_l} = -w_{\lambda'_l}$ ,  $l = 1, \dots, d'$  and  $\zeta_{\lambda''_s} = w_{\lambda''_s}$ ,  $s = 1, \dots, d''$ . Also, denote

$$\mathbf{D}_{(\Lambda')} = \{\zeta \in \mathbf{D} \mid \operatorname{Re} \zeta_{\lambda'_l} \geq 0, \quad l = 1, \dots, d', \quad \text{and} \quad \operatorname{Re} \zeta_{\lambda''_s} \leq 0, \quad s = 1, \dots, d''\}$$

(note that  $\mathbf{D}_{(\emptyset)}$  consists of all  $w \in \mathbf{D}$  such that the coordinates of  $w$  have non-positive real parts).

For  $f \in \mathcal{S}(\mathbb{R}_+^d)$  let  $a_n = \langle f, \mathcal{L}_n \rangle$ ,  $n \in \mathbb{N}_0^d$ . Then, Proposition 5.3 implies  $\mathcal{F}_\mathbf{D} f(w) = (\mathbf{1} - w)^{\mathbf{1}} \sum_{n \in \mathbb{N}_0^d} a_n w^n$ ,  $w \in \mathbf{D}$ . As  $\langle \mathcal{H}_0^{(\Lambda')} f, \mathcal{L}_n \rangle = (-1)^{n_{\lambda'_1} + \dots + n_{\lambda'_{d'}}} a_n$  and

$$\mathcal{F}_\mathbf{D}(\mathcal{H}_0^{(\Lambda')} f)(w) = (\mathbf{1} - w)^{\mathbf{1}} \sum_{n \in \mathbb{N}_0^d} (-1)^{n_{\lambda'_1} + \dots + n_{\lambda'_{d'}}} a_n w^n, \quad w \in \mathbf{D}.$$

Hence,

$$\mathcal{F}_\mathbf{D}(\mathcal{H}_0^{(\Lambda')} f)(\tilde{\Lambda}' w) = \left( \prod_{l=1}^{d'} \frac{1 + w_{\lambda'_l}}{1 - w_{\lambda'_l}} \right) \cdot (\mathbf{1} - w)^{\mathbf{1}} \sum_{n \in \mathbb{N}_0^d} a_n w^n, \quad w \in \mathbf{D}.$$

Thus

$$(23) \quad \mathcal{F}_\mathbf{D} f(w) = \left( \prod_{l=1}^{d'} \frac{1 - w_{\lambda'_l}}{1 + w_{\lambda'_l}} \right) \mathcal{F}_\mathbf{D}(\mathcal{H}_0^{(\Lambda')} f)(\tilde{\Lambda}' w), \quad w \in \mathbf{D}, \quad f \in \mathcal{S}(\mathbb{R}_+^d).$$

Let  $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$ . Since  $\mathcal{H}_0^{(\Lambda')} f \in G_\alpha^\alpha(\mathbb{R}_+^d)$  (cf. Corollary 4.7), Proposition 5.5 implies the existence of  $A, C > 0$  such that

$$(24) \quad \left| D^n \mathcal{F}_\mathbf{D}(\mathcal{H}_0^{(\Lambda')} f)(w) \right| \leq CA^{|n|} n^{\alpha n}, \quad \forall n \in \mathbb{N}_0^d, \quad \forall w \in \mathbf{D}_{(\emptyset)}, \quad \forall \Lambda' \subseteq \{1, \dots, d\}.$$

Observe that for  $w \in \mathbf{D}_{(\emptyset)}$ , (22) holds by Proposition 5.5. To prove (22) for  $w \in \mathbf{D}_{(\Lambda')}$  when  $\emptyset \neq \Lambda' \subseteq \{1, \dots, d\}$ , we need an estimate for the derivatives of the function  $\zeta \mapsto (1 - \zeta)/(1 + \zeta)$ ,  $\{\zeta \in \mathbb{C} \mid |\zeta| < 1\} \rightarrow \mathbb{C}$  when  $\operatorname{Re} \zeta \geq 0$ . Since  $(1 - \zeta)/(1 + \zeta) = 2/(1 + \zeta) - 1$  and  $|1 + \zeta| \geq 1$  when  $\operatorname{Re} \zeta \geq 0$ , for  $j \in \mathbb{N}$  we have

$$(25) \quad \left| \frac{d^j}{d\zeta^j} \left( \frac{1 - \zeta}{1 + \zeta} \right) \right| = \frac{2j!}{|1 + \zeta|^{j+1}} \leq 2j!, \quad \text{when } |\zeta| < 1 \text{ and } \operatorname{Re} \zeta \geq 0.$$

Clearly, (25) also holds for  $j = 0$ . Now, observe that  $\tilde{\Lambda}'(\mathbf{D}_{(\Lambda')}) = \mathbf{D}_{(\emptyset)}$ . Hence, for  $w \in \mathbf{D}_{(\Lambda')}$ , (23), (24) and (25) imply

$$|D^n \mathcal{F}_\mathbf{D} f(w)| \leq \sum_{m_{\Lambda'} \leq n_{\Lambda'}} \binom{n_{\Lambda'}}{m_{\Lambda'}} 2^{d'} m_{\Lambda'}! \left| D_{w_{\Lambda'}}^{n_{\Lambda'} - m_{\Lambda'}} D_{w_{\Lambda''}}^{n_{\Lambda''}} \mathcal{F}_\mathbf{D}(\mathcal{H}_0^{(\Lambda')} f)(\tilde{\Lambda}' w) \right|$$

$$\begin{aligned}
&\leq C_1 \sum_{m_{\Lambda'} \leq n_{\Lambda'}} \binom{n_{\Lambda'}}{m_{\Lambda'}} m_{\Lambda'}^{\alpha m_{\Lambda'}} A^{|n|-|m_{\Lambda'}|} (n_{\Lambda'} - m_{\Lambda'})^{\alpha(n_{\Lambda'} - m_{\Lambda'})} n_{\Lambda''}^{\alpha n_{\Lambda''}} \\
&\leq C_1 (2A)^{|n|} n^{\alpha n},
\end{aligned}$$

which completes the proof.  $\square$

Now, Proposition 5.1, Proposition 5.6, Proposition 5.3 and Proposition 5.4 give the main result of this section:

**Theorem 5.7.** ([8, Theorem 3.6], for  $d=1$ ) *Let  $\alpha \geq 1$ . For  $f \in L^2(\mathbb{R}_+^d)$  let  $a_n = \int_{\mathbb{R}_+^d} f(t) \mathcal{L}_n(t) dt$ ,  $n \in \mathbb{N}_0^d$ . The following conditions are equivalent:*

- (i) *There exist  $c > 0$  and  $a > 1$  such that  $|a_n| \leq ca^{-|n|^{1/\alpha}}$  for  $n \in \mathbb{N}_0^d$ .*
- (ii)  *$f \in G_\alpha^\alpha(\mathbb{R}_+^d)$ .*
- (iii) *There exist  $C, A > 0$  such that*

$$|D^p \mathcal{F}_\mathbf{D}(f)(w)| \leq CA^{|p|} p^{\alpha p} \quad \text{for } p \in \mathbb{N}_0^d \quad \text{and } w \in \mathbf{D}$$

$$\text{and } \lim_{w \rightarrow 1} \mathcal{F}_\mathbf{D}(f)(w) = 0.$$

*Conversely, given a sequence  $\{a_n\}_{n \in \mathbb{N}_0^d}$  satisfying condition (i) or given  $F \in \mathcal{O}(\mathbf{D})$  of the form  $F(w) = (1-w)^1 \sum_n a_n w^n$  with  $a_n \rightarrow 0$  as  $|n| \rightarrow \infty$  which satisfies (17), there exists  $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$  such that  $a_n = \int_{\mathbb{R}_+^d} f(t) \mathcal{L}_n(t) dt$  and  $\mathcal{F}_\mathbf{D}(f)(w) = F(w)$  for  $w \in \mathbf{D}$ .*

## 6. TOPOLOGICAL PROPERTIES OF $G_\alpha^\alpha(\mathbb{R}_+^d)$ , $\alpha \geq 1$ . THE KERNEL THEOREMS

As we shall see, we gain deep insights into the topological structure of  $G_\alpha^\alpha(\mathbb{R}_+^d)$ ,  $\alpha \geq 1$  by Theorem 5.7. Let  $\iota : G_\alpha^\alpha(\mathbb{R}_+^d) \rightarrow \mathfrak{s}^\alpha$ ,  $\iota(f) = \{\langle f, \mathcal{L}_n \rangle\}_{n \in \mathbb{N}_0^d}$ . Theorem 5.7 proves that  $\iota$  is a well defined bijection.

**Theorem 6.1.** *Let  $\alpha \geq 1$ . The mapping  $\iota : G_\alpha^\alpha(\mathbb{R}_+^d) \rightarrow \mathfrak{s}^\alpha$ ,  $\iota(f) = \{\langle f, \mathcal{L}_n \rangle\}_{n \in \mathbb{N}_0^d}$ , is a topological isomorphism between  $G_\alpha^\alpha(\mathbb{R}_+^d)$  and  $\mathfrak{s}^\alpha$ . In particular,  $G_\alpha^\alpha(\mathbb{R}_+^d)$  is a  $(DFN)$ -space and  $(G_\alpha^\alpha(\mathbb{R}_+^d))'$  is an  $(FN)$ -space.*

*For each  $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$ ,  $\sum_{n \in \mathbb{N}_0^d} \langle f, \mathcal{L}_n \rangle \mathcal{L}_n$  is summable to  $f$  in  $G_\alpha^\alpha(\mathbb{R}_+^d)$ .*

*Proof.* If we consider  $\iota$  as a linear mapping from  $G_\alpha^\alpha(\mathbb{R}_+^d)$  into  $\mathfrak{s}$  ( $\mathfrak{s}^\alpha$  is canonically injected into  $\mathfrak{s}$ ) then  $\iota$  is continuous since it decomposes as  $G_\alpha^\alpha(\mathbb{R}_+^d) \rightarrow \mathcal{S}(\mathbb{R}_+^d) \xrightarrow{f \mapsto \{\langle f, \mathcal{L}_n \rangle\}_n} \mathfrak{s}$ , where the first mapping is the canonical inclusion. Hence,  $\iota$  has a closed graph in  $G_\alpha^\alpha(\mathbb{R}_+^d) \times \mathfrak{s}$ . Since the range of  $\iota$  is in  $\mathfrak{s}^\alpha$  and  $\mathfrak{s}^\alpha$  is continuously injected into  $\mathfrak{s}$ , the graph of  $\iota$  is closed in  $G_\alpha^\alpha(\mathbb{R}_+^d) \times \mathfrak{s}^\alpha$ .  $G_\alpha^\alpha(\mathbb{R}_+^d)$  is injective inductive limit of  $(F)$ -spaces. For this reason,  $G_\alpha^\alpha(\mathbb{R}_+^d)$  is ultrabornological (cf. [15, Theorem 7, p. 72]; every  $(F)$ -space is ultrabornological). Moreover,  $\mathfrak{s}^\alpha$  is a webbed space of De Wilde (see [15, Theorem 11, p. 64]). Hence, the closed graph theorem of De Wilde (see [15, Theorem 2, p. 57]) implies that  $\iota : G_\alpha^\alpha(\mathbb{R}_+^d) \rightarrow \mathfrak{s}^\alpha$  is continuous. Also,  $\mathfrak{s}^\alpha$  is ultrabornological since it is bornological and complete and  $G_\alpha^\alpha(\mathbb{R}_+^d)$  is a webbed space of De Wilde (cf. [15, Theorem 8, p. 63]; every  $(F)$ -space is a webbed space of De Wilde). The mapping  $\iota^{-1} : \mathfrak{s}^\alpha \rightarrow G_\alpha^\alpha(\mathbb{R}_+^d)$ , which has a closed graph, is continuous by the De Wilde closed graph theorem (see [15, Theorem 2, p. 57]). Now, Proposition 2.1 implies that  $G_\alpha^\alpha(\mathbb{R}_+^d)$  is a  $(DFN)$ -space and  $(G_\alpha^\alpha(\mathbb{R}_+^d))'$  is an  $(FN)$ -space.

Given  $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$ , let  $a_n = \langle f, \mathcal{L}_n \rangle$ . For each finite  $\Phi \subseteq \mathbb{N}_0^d$ , denote  $f_\Phi = \sum_{n \in \Phi} a_n \mathcal{L}_n \in G_\alpha^\alpha(\mathbb{R}_+^d)$  (since  $\mathcal{L}_n \in G_\alpha^\alpha(\mathbb{R}_+^d)$ ). Let  $a > 1$  be such that  $\iota(f) \in \mathfrak{s}^{\alpha, a}$ . Fix  $1 < a' < a$ . One easily verifies that for each  $\varepsilon > 0$  there exists finite  $\Phi_0 \subseteq \mathbb{N}_0^d$  such that for each finite  $\Phi \subseteq \mathbb{N}_0^d$ , satisfying  $\Phi_0 \subseteq \Phi$ , we have  $\|\iota(f) - \iota(f_\Phi)\|_{\mathfrak{s}^{\alpha, a'}} \leq \varepsilon$ . Since  $\iota$  is an isomorphism this implies that for each neighbourhood of zero  $V \subseteq G_\alpha^\alpha(\mathbb{R}_+^d)$  there exists finite  $\Phi_0 \subseteq \mathbb{N}_0^d$  such that for finite  $\Phi \supseteq \Phi_0$  we have  $f - f_\Phi \in V$ , i.e.  $\sum_{n \in \mathbb{N}_0^d} a_n \mathcal{L}_n$  is summable to  $f$  in  $G_\alpha^\alpha(\mathbb{R}_+^d)$ .  $\square$

**Theorem 6.2.** *Let  $\alpha \geq 1$ . The mapping  $\tilde{\iota} : (G_\alpha^\alpha(\mathbb{R}_+^d))' \rightarrow (\mathfrak{s}^\alpha)'$ ,  $\tilde{\iota}(T) = \{\langle T, \mathcal{L}_n \rangle\}_{n \in \mathbb{N}_0^d}$ , is a topological isomorphism. Moreover,  $\sum_{n \in \mathbb{N}_0^d} \langle T, \mathcal{L}_n \rangle \mathcal{L}_n$  is summable to  $T$  in  $(G_\alpha^\alpha(\mathbb{R}_+^d))'$ .*

*Proof.* By Theorem 6.1, both the transpose of  $\iota$ ,  ${}^t\iota : (\mathfrak{s}^\alpha)' \rightarrow (G_\alpha^\alpha(\mathbb{R}_+^d))'$ , and its inverse  $({}^t\iota)^{-1} : (G_\alpha^\alpha(\mathbb{R}_+^d))' \rightarrow (\mathfrak{s}^\alpha)'$  are topological isomorphisms. For  $T \in (G_\alpha^\alpha(\mathbb{R}_+^d))'$ , let  $\{b_n\}_n = ({}^t\iota)^{-1}(T)$ . Then

$$\langle T, \mathcal{L}_n \rangle = \langle {}^t\iota(\{b_n\}_n), \mathcal{L}_n \rangle = \langle \{b_n\}_n, \iota(\mathcal{L}_n) \rangle = b_n.$$

Thus  $\{\langle T, \mathcal{L}_n \rangle\}_n = \{b_n\}_n = ({}^t\iota)^{-1}(T) \in (\mathfrak{s}^\alpha)'$ . Hence  $\tilde{\iota}$  is in fact a topological isomorphism  $({}^t\iota)^{-1} : (G_\alpha^\alpha(\mathbb{R}_+^d))' \rightarrow (\mathfrak{s}^\alpha)'$ . Now, by the similar approach as above, one proves that  $\sum_{n \in \mathbb{N}_0^d} \langle T, \mathcal{L}_n \rangle \mathcal{L}_n$  is summable to  $T$  in  $(G_\alpha^\alpha(\mathbb{R}_+^d))'$ .  $\square$

For  $T \in (G_\alpha^\alpha(\mathbb{R}_+^d))'$ , by Lemma 5.2,  $\mathcal{F}_\mathbf{D} T \in \mathcal{O}(\mathbf{D})$ . Since  $\sum_n \langle T, \mathcal{L}_n \rangle \mathcal{L}_n$  is summable to  $T$  in  $(G_\alpha^\alpha(\mathbb{R}_+^d))'$ , by the same method as in the proof of Proposition 5.3, one proves the following result.

**Proposition 6.3.** *Let  $T \in (G_\alpha^\alpha(\mathbb{R}_+^d))'$ ,  $\alpha \geq 1$  and  $b_n = \langle T, \mathcal{L}_n \rangle$ ,  $n \in \mathbb{N}_0^d$ . Then,*

$$\mathcal{F}_\mathbf{D}(T)(w) = \prod_{j=1}^d (1 - w_j) \sum_{n \in \mathbb{N}_0^d} b_n w^n, \quad w \in \mathbf{D}.$$

*In particular, if  $\mathcal{F}_\mathbf{D} T = 0$  then  $T = 0$ .*

Now, we state the kernel theorems:

**Theorem 6.4.** *Let  $\alpha \geq 1$ . We have the following canonical isomorphism:*

$$G_\alpha^\alpha(\mathbb{R}_+^{d_1}) \hat{\otimes} G_\alpha^\alpha(\mathbb{R}_+^{d_2}) \cong G_\alpha^\alpha(\mathbb{R}_+^{d_1+d_2}), \quad (G_\alpha^\alpha(\mathbb{R}_+^{d_1}))' \hat{\otimes} (G_\alpha^\alpha(\mathbb{R}_+^{d_2}))' \cong (G_\alpha^\alpha(\mathbb{R}_+^{d_1+d_2}))'.$$

*Proof.* For simplicity, put  $d = d_1 + d_2$ . Let  $\mathfrak{s}_{d_1}^\alpha$ ,  $\mathfrak{s}_{d_2}^\alpha$  and  $\mathfrak{s}^\alpha$  be the  $d_1$ -dimensional, the  $d_2$ -dimensional and the  $d$ -dimensional variant of the space  $\mathfrak{s}^\alpha$ , respectively. Firstly, we prove that  $(\mathfrak{s}_{d_1}^\alpha)' \hat{\otimes} (\mathfrak{s}_{d_2}^\alpha)' \cong (\mathfrak{s}^\alpha)'$ , where an isomorphism is given by the extension of the canonical inclusion  $(\mathfrak{s}_{d_1}^\alpha)' \otimes (\mathfrak{s}_{d_2}^\alpha)' \rightarrow (\mathfrak{s}^\alpha)'$ ,  $\{u_n\}_{n \in \mathbb{N}_0^{d_1}} \otimes \{v_m\}_{m \in \mathbb{N}_0^{d_2}} \mapsto \{u_n v_m\}_{(n,m) \in \mathbb{N}_0^d}$ . Observe that the mapping

$$\left( \{u_n\}_{n \in \mathbb{N}_0^{d_1}}, \{v_m\}_{m \in \mathbb{N}_0^{d_2}} \right) \mapsto \{u_n v_m\}_{(n,m) \in \mathbb{N}_0^d}, (\mathfrak{s}_{d_1}^\alpha)' \times (\mathfrak{s}_{d_2}^\alpha)' \rightarrow (\mathfrak{s}^\alpha)'$$

is continuous. Hence, the  $\pi$  topology on  $(\mathfrak{s}_{d_1}^\alpha)' \otimes (\mathfrak{s}_{d_2}^\alpha)'$  is stronger than the induced one from  $(\mathfrak{s}^\alpha)'$ . Let  $A$  and  $B$  be the equicontinuous subsets of  $((\mathfrak{s}_{d_1}^\alpha)')' = \mathfrak{s}_{d_1}^\alpha$  and

$((\mathfrak{s}_{d_2}^\alpha)')' = \mathfrak{s}_{d_2}^\alpha$ , respectively  $(\mathfrak{s}^\alpha)$  is reflexive since it is a  $(DFN)$ -space). Hence, there exist  $C > 0$  and  $r > 1$  such that

$$\begin{aligned} |\langle \{u_n\}_{n \in \mathbb{N}_0^{d_1}}, \{a_n\}_{n \in \mathbb{N}_0^{d_1}} \rangle| &\leq C \sum_{n \in \mathbb{N}_0^{d_1}} |u_n| r^{-|n|^{1/\alpha}}, \quad \forall \{a_n\}_{n \in \mathbb{N}_0^{d_1}} \in A, \quad \forall \{u_n\}_{n \in \mathbb{N}_0^{d_1}} \in (\mathfrak{s}_{d_1}^\alpha)' \\ |\langle \{v_m\}_{m \in \mathbb{N}_0^{d_2}}, \{b_m\}_{m \in \mathbb{N}_0^{d_2}} \rangle| &\leq C \sum_{m \in \mathbb{N}_0^{d_2}} |v_m| r^{-|m|^{1/\alpha}}, \quad \{b_m\}_{m \in \mathbb{N}_0^{d_2}} \in B, \quad \{v_m\}_{m \in \mathbb{N}_0^{d_2}} \in (\mathfrak{s}_{d_2}^\alpha)'. \end{aligned}$$

Let  $\{\chi(n, m)\}_{(n, m) \in \mathbb{N}_0^d} = \sum_{j=1}^l \{u_n^{(j)}\}_{n \in \mathbb{N}_0^{d_1}} \otimes \{v_m^{(j)}\}_{m \in \mathbb{N}_0^{d_2}} \in (\mathfrak{s}_{d_1}^\alpha)' \otimes (\mathfrak{s}_{d_2}^\alpha)'$ . Then, for  $\{a_n\}_{n \in \mathbb{N}_0^{d_1}} \in A$  and  $\{b_m\}_{m \in \mathbb{N}_0^{d_2}} \in B$ , we have

$$\begin{aligned} &|\langle \{\chi(n, m)\}_{(n, m)}, \{a_n\}_n \otimes \{b_m\}_m \rangle| \\ &= \left| \left\langle \sum_{j=1}^l \{u_n^{(j)}\}_n \langle \{v_m^{(j)}\}_m, \{b_m\}_m \rangle, \{a_n\}_n \right\rangle \right| = \left| \left\langle \left\{ \sum_{j=1}^l \langle \{v_m^{(j)}\}_m, \{b_m\}_m \rangle u_n^{(j)} \right\}_n, \{a_n\}_n \right\rangle \right| \\ &\leq C \sum_{n \in \mathbb{N}_0^{d_1}} \left| \sum_{j=1}^l \langle \{v_m^{(j)}\}_m, \{b_m\}_m \rangle u_n^{(j)} \right| r^{-|n|^{1/\alpha}} \\ &= C \sum_{n \in \mathbb{N}_0^{d_1}} \left| \left\langle \left\{ \sum_{j=1}^l u_n^{(j)} v_m^{(j)} \right\}_m, \{b_m\}_m \right\rangle \right| r^{-|n|^{1/\alpha}} \leq C^2 \sum_{(n, m) \in \mathbb{N}^d} \left| \sum_{j=1}^l u_n^{(j)} v_m^{(j)} \right| r^{-|n|^{1/\alpha} - |m|^{1/\alpha}} \\ &\leq C^2 \|\{\chi(n, m)\}_{(n, m)}\|_{(\mathfrak{s}^\alpha)', r}. \end{aligned}$$

We can conclude that the  $\epsilon$  topology on  $(\mathfrak{s}_{d_1}^\alpha)' \otimes (\mathfrak{s}_{d_2}^\alpha)'$  is weaker than the induced one from  $(\mathfrak{s}^\alpha)'$ . Since  $(\mathfrak{s}^\alpha)'$  is nuclear, these topologies are identical. Clearly,  $(\mathfrak{s}_{d_1}^\alpha)' \otimes (\mathfrak{s}_{d_2}^\alpha)'$  is dense in  $(\mathfrak{s}_d^\alpha)'$ . Hence, we proved the desired topological isomorphism. As all spaces in consideration are  $(FN)$ -spaces, by duality we have  $\mathfrak{s}_{d_1}^\alpha \hat{\otimes} \mathfrak{s}_{d_2}^\alpha \cong \mathfrak{s}^\alpha$ . Note that the isomorphism is in fact the extension of the canonical inclusion  $\kappa : \mathfrak{s}_{d_1}^\alpha \otimes \mathfrak{s}_{d_2}^\alpha \rightarrow \mathfrak{s}^\alpha$ ,  $\kappa(\{a_n\}_n \otimes \{b_m\}_m) = \{a_n b_m\}_{(n, m)}$ . Now observe that the diagram

$$\begin{array}{ccc} \mathfrak{s}_{d_1}^\alpha \otimes \mathfrak{s}_{d_2}^\alpha & \xrightarrow{\kappa} & \mathfrak{s}^\alpha \\ \uparrow \iota \otimes \iota & & \uparrow \iota \\ G_\alpha^\alpha(\mathbb{R}_+^{d_1}) \otimes G_\alpha^\alpha(\mathbb{R}_+^{d_2}) & \longrightarrow & G_\alpha^\alpha(\mathbb{R}_+^d) \end{array}$$

commutes, where the bottom horizontal line is the canonical inclusion  $f \otimes g(x, y) \mapsto f(x)g(y)$ . Since  $\kappa$  extends to an isomorphism, by Theorem 6.1, it follows that the canonical inclusion  $G_\alpha^\alpha(\mathbb{R}_+^{d_1}) \otimes G_\alpha^\alpha(\mathbb{R}_+^{d_2}) \rightarrow G_\alpha^\alpha(\mathbb{R}_+^d)$  is continuous and it extends to an isomorphism  $G_\alpha^\alpha(\mathbb{R}_+^{d_1}) \hat{\otimes} G_\alpha^\alpha(\mathbb{R}_+^{d_2}) \cong G_\alpha^\alpha(\mathbb{R}_+^d)$ . The assertion  $(G_\alpha^\alpha(\mathbb{R}_+^{d_1}))' \hat{\otimes} (G_\alpha^\alpha(\mathbb{R}_+^{d_2}))' \cong (G_\alpha^\alpha(\mathbb{R}_+^d))'$  can be obtained by the duality of an isomorphism  $G_\alpha^\alpha(\mathbb{R}_+^{d_1}) \hat{\otimes} G_\alpha^\alpha(\mathbb{R}_+^{d_2}) \cong G_\alpha^\alpha(\mathbb{R}_+^d)$  since  $G_\alpha^\alpha(\mathbb{R}_+^d)$  is a  $(DFN)$ -space.  $\square$   $\square$

## 7. WEYL PSEUDO-DIFFERENTIAL OPERATORS WITH RADIAL SYMBOLS FROM THE $G$ -TYPE SPACES AND THEIR DUALS

Concerning pseudo-differential operators, especially the Weyl calculus, we refer to the standard books [19] and [24], for example.

Let  $f, g \in \mathcal{S}(\mathbb{R}^d)$ . Then the function  $W(f, g)$  defined on  $\mathbb{R}^{2d}$  by

$$W(f, g)(x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\xi \cdot p} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)} dp, \quad x, \xi \in \mathbb{R}^d$$

is called the Wigner transform of  $f$  and  $g$ . The bilinear mapping  $(f, g) \mapsto W(f, \bar{g})$ ,  $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^{2d})$  is continuous and (cf. [27, Corollary 3.4]) can be extended uniquely to a bilinear operator  $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^{2d})$  such that

$$\|W(f, g)\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}, \quad f, g \in L^2(\mathbb{R}^d).$$

Let  $f, g \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ ,  $\alpha \geq 1/2$ . By [25, Theorem 3.8, p. 179],  $W(f, g) \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d})$ . Moreover, we have the following proposition.

**Proposition 7.1.** *A bilinear mapping  $(f, g) \mapsto W(f, \bar{g})$ ,  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \times \mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \rightarrow \mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d})$ , is continuous.*

*Proof.* Fix  $g \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ . If we consider a mapping  $f \mapsto W(f, \bar{g})$  as a mapping from  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  into  $\mathcal{S}(\mathbb{R}^{2d})$  it is continuous since it decomposes as  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \xrightarrow{f \mapsto W(f, \bar{g})} \mathcal{S}(\mathbb{R}^{2d})$ , where the first mapping is the canonical inclusion. Hence, its graph is closed in  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^{2d})$ . Since its image is in  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d})$ , its graph is closed in  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \times \mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d})$ . As  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  is a  $(DFS)$ -space it is an ultrabornological and webbed space of De Wilde (cf. [15, Theorem 11, p. 64]). Now, the De Wilde closed graph theorem (see [15, Theorem 2, p. 57]) implies its continuity. Similarly, for each fixed  $f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ , the mapping  $g \mapsto W(f, \bar{g})$ ,  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \rightarrow \mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d})$  is continuous. Thus the bilinear mapping  $(f, g) \mapsto W(f, \bar{g})$ ,  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \times \mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \rightarrow \mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d})$ , is separately continuous and hence continuous since  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  is barrelled  $(DF)$ -space (cf. [15, Theorem 11, p. 161]).  $\square$   $\square$

Recall, for  $\sigma \in \mathcal{S}(\mathbb{R}^{2d})$  the Weyl pseudo-differential operator with symbol  $\sigma$  is defined by

$$(26) \quad (W_\sigma f)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi} \sigma\left(\frac{x+y}{2}, \xi\right) f(y) dy d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

and can be extended on  $(\mathcal{S}(\mathbb{R}^d))'$  as a linear and continuous operator from  $(\mathcal{S}(\mathbb{R}^d))'$  into itself. Let  $\alpha \geq 1/2$ . The Weyl pseudo-differential operator with a symbol  $\sigma \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d}))'$  defined by

$$(27) \quad (W_\sigma f)(g) = (2\pi)^{-d/2} \langle \sigma, W(f, \bar{g}) \rangle$$

is a continuous and linear mapping from  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  into  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ .

Our goal is to analyse the Weyl pseudo-differential operator on  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  and  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ ,  $\alpha \geq 1/2$ , when its symbol originates from  $(G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$ ,  $\alpha \geq 1/2$ .

Throughout the rest of this section, we denote by  $v$  the mapping  $\mathbb{R}^{2d} \rightarrow \overline{\mathbb{R}_+^d}$ ,  $(x, \xi) \mapsto v(x, \xi) = (x_1^2 + \xi_1^2, \dots, x_d^2 + \xi_d^2)$ .

**Proposition 7.2.** *Let  $\sigma \in \mathcal{S}(\mathbb{R}_+^d)$ . Then  $\tilde{\sigma}(x, \xi) = \sigma \circ v(x, \xi) \in \mathcal{S}(\mathbb{R}^{2d})$ . Moreover, the mapping  $\sigma \mapsto \tilde{\sigma} = \sigma \circ v, \mathcal{S}(\mathbb{R}_+^d) \rightarrow \mathcal{S}(\mathbb{R}^{2d})$ , is continuous.*

*Proof.* Fix  $j \in \mathbb{N}$ . For  $p, q \in \mathbb{N}_0^d$ ,  $|p| \leq j$  and  $|q| \leq j$  observe that  $D_x^p D_\xi^q \tilde{\sigma}(x, \xi)$  is a finite sum of the form  $P(x, \xi) D_x^{p'} D_\xi^{q'} \sigma(v(x, \xi))$ , where  $P(x, \xi)$  are polynomials in  $(x, \xi)$  of degree at most  $|p| + |q|$  which do not depend on  $\sigma$  (they only depend on the derivatives of  $v$ ) and  $p', q' \in \mathbb{N}_0^d$  are such that  $p' \leq p$  and  $q' \leq q$ . Moreover, observe that the number of such terms that appear in  $D_x^p D_\xi^q \tilde{\sigma}(x, \xi)$  depend only on  $p$  and  $q$  (and not on  $\sigma$ ). For  $p'', q'' \in \mathbb{N}_0^d$  we also have  $|x^{p''} \xi^{q''}| \leq |x|^{|p''|} |\xi|^{|q''|} \leq (|x|^2 + |\xi|^2)^{(|p''| + |q''|)/2}$ . Thus,

$$\sup_{\substack{|p''| \leq j \\ |q''| \leq j}} \sup_{|p| \leq j} \sup_{(x, \xi) \in \mathbb{R}^{2d}} |x^{p''} \xi^{q''} D_x^p D_\xi^q \tilde{\sigma}(x, \xi)| \leq C \sup_{\substack{|n| \leq 2j \\ |m| \leq 2j}} \sup_{t \in \mathbb{R}_+^d} |t^m D^n \sigma(t)|.$$

Hence,  $\tilde{\sigma} \in \mathcal{S}(\mathbb{R}^{2d})$  and the mapping  $\sigma \mapsto \tilde{\sigma} = \sigma \circ v, \mathcal{S}(\mathbb{R}_+^d) \rightarrow \mathcal{S}(\mathbb{R}^{2d})$  is continuous.  $\square$

Let  $\alpha \geq 1/2$  and  $\sigma(\rho) \in G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d)$ . Denote by  $\sigma_0(\rho) = \sigma(2\rho)$ ,  $\rho \in \mathbb{R}_+^d$ . Then the functions  $\tilde{\sigma}$  and  $\tilde{\sigma}_0$  defined by

$$(28) \quad \tilde{\sigma}(x, \xi) = \sigma \circ v(x, \xi), \quad \tilde{\sigma}_0(x, \xi) = \sigma_0 \circ v(x, \xi), \quad (x, \xi) \in \mathbb{R}^{2d}$$

belong to  $\mathcal{S}(\mathbb{R}^{2d})$  (see Proposition 7.2). Hence, the Weyl pseudo-differential operator with a symbol  $\tilde{\sigma}_0$  is a continuous mapping  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \rightarrow (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ .

**Theorem 7.3.** *Let  $\alpha \geq 1/2$  and  $\sigma(\rho) \in G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d)$ . Denote by  $\sigma_0(\rho) = \sigma(2\rho)$ ,  $\rho \in \mathbb{R}_+^d$ . Let  $\tilde{\sigma}, \tilde{\sigma}_0 \in \mathcal{S}(\mathbb{R}^{2d})$  be the functions defined in (28). Then the Weyl pseudo-differential operator  $W_{\tilde{\sigma}_0}$  is a continuous operator  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \rightarrow \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  and it extends to a continuous mapping  $W_{\tilde{\sigma}_0} : (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))' \rightarrow \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ . If  $f, g \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$  and*

$$f_k = \langle f, h_k \rangle, g_k = \langle g, h_k \rangle \text{ and } \sigma_k = (2\pi)^{d/2} (-1)^{|k|} 2^{-d} \int_{\mathbb{R}_+^d} \sigma(\rho) \mathcal{L}_k(\rho) d\rho,$$

then  $(W_{\tilde{\sigma}_0} f)(g) = (2\pi)^{-d/2} \sum_{k \in \mathbb{N}_0^d} f_k g_k \sigma_k$ . Moreover, if  $\sigma_{0,j}(\eta) \rightarrow \sigma_0(\eta)$  in  $G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d)$  as  $j \rightarrow \infty$  then  $W_{\tilde{\sigma}_{0,j}} \rightarrow W_{\tilde{\sigma}_0}$  in the strong topology of  $\mathcal{L}((\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))', \mathcal{S}_\alpha^\alpha(\mathbb{R}^d))$ .

*Proof.* First we compute the Weyl pseudo-differential transform  $W_{\tilde{\sigma}_0}$  of  $f, g \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ . Since  $\sum_{n \in \mathbb{N}_0^d} f_n h_n$  and  $\sum_{n \in \mathbb{N}_0^d} g_n h_n$  converge absolutely to  $f$  and  $g$  in  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  respectively (cf. Proposition 2.3) and the mapping  $(\varphi, \psi) \mapsto W(\varphi, \bar{\psi}), \mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \times \mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \rightarrow \mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d})$ , is continuous (see Proposition 7.1), we conclude  $W(f, \bar{g}) = \sum_{(m,k) \in \mathbb{N}_0^{2d}} f_m g_k W(h_m, h_k)$ , where the sum converges absolutely in  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d})$ . As  $\tilde{\sigma}_0 \in \mathcal{S}(\mathbb{R}^{2d}) \subseteq (\mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d}))'$ , we have

$$(29) \quad (W_{\tilde{\sigma}_0} f)(g) = (2\pi)^{-d/2} \sum_{(m,k) \in \mathbb{N}_0^{2d}} f_m g_k \langle \tilde{\sigma}_0, \psi_{m,k} \rangle,$$

where  $\psi_{m,k} = W(h_m, h_k)$ . Clearly,  $\psi_{m,k} = \prod_{r=1}^d \psi_{m_r, k_r}$ , where  $\psi_{m_r, k_r} = W(h_{m_r}, h_{k_r})$ . Using [27, Theorem 24.1] and denoting  $\eta_r = x_r + i\xi_r \in \mathbb{C}$ , we have

$$\psi_{m_r, k_r}(x_r, \xi_r) = 2(-1)^{k_r} (2\pi)^{-1/2} \left( \frac{k_r!}{m_r!} \right)^{1/2} (\sqrt{2})^{m_r - k_r} (\overline{\eta_r})^{m_r - k_r} L_{k_r}^{m_r - k_r}(2|\eta_r|^2) e^{-|\eta_r|^2}, \quad m_r \geq k_r,$$

$$\psi_{m_r, k_r}(x_r, \xi_r) = 2(-1)^{m_r} (2\pi)^{-1/2} \left( \frac{m_r!}{k_r!} \right)^{1/2} (\sqrt{2})^{k_r - m_r} \eta_r^{k_r - m_r} L_{m_r}^{k_r - m_r}(2|\eta_r|^2) e^{-|\eta_r|^2}, \text{ if } k_r \geq m_r.$$

In terms of polar coordinates the integral  $\langle \tilde{\sigma}_0, \psi_{m,k} \rangle = \int_{\mathbb{R}^{2d}} \sigma_0(v(x, \xi)) \psi_{m,k}(x, \xi) dx d\xi$  is

$$\langle \tilde{\sigma}_0, \psi_{m,k} \rangle = C_{m,k} \prod_{r=1}^d \int_{-\pi}^{\pi} e^{-i(m_r - k_r)\theta_r} d\theta_r.$$

Thus  $\langle \tilde{\sigma}_0, \psi_{m,k} \rangle = 0$  when  $m \neq k$ . Moreover,

$$\begin{aligned} \langle \tilde{\sigma}_0, \psi_{k,k} \rangle &= (2\pi)^{d/2} (-1)^{|k|} 2^d \int_{\mathbb{R}_+^d} \sigma(2\rho_1^2, \dots, 2\rho_d^2) L_k(2\rho_1^2, \dots, 2\rho_d^2) e^{-|\rho|^2} \rho^1 d\rho \\ &= (2\pi)^{d/2} (-1)^{|k|} 2^{-d} \int_{\mathbb{R}_+^d} \sigma(y) \mathcal{L}_k(y) dy = \sigma_k. \end{aligned}$$

By (29), we obtain

$$(30) \quad (W_{\tilde{\sigma}_0} f)(g) = (2\pi)^{-d/2} \sum_{k \in \mathbb{N}_0^d} f_k g_k \sigma_k$$

and the series converges absolutely since  $\{f_n\}_n, \{g_n\}_n, \{\sigma_n\}_n \in \mathfrak{s}^{2\alpha}$  ( $f, g \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ ,  $\sigma \in G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d)$ ). Let now  $f, g \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ . Define  $(W_{\tilde{\sigma}_0} f)(g) = (2\pi)^{-d/2} \sum_{n \in \mathbb{N}_0^d} f_n g_n \sigma_n$ . Observe that the series converges absolutely since  $\{f_n\}_{n \in \mathbb{N}_0^d}, \{g_n\}_{n \in \mathbb{N}_0^d} \in (\mathfrak{s}^{2\alpha})'$  and  $\{\sigma_n\}_{n \in \mathbb{N}_0^d} \in \mathfrak{s}^{2\alpha}$  ( $\sigma \in G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d)$ ; cf. Theorem 6.1). Thus, if we fix  $f \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ , the mapping  $g \mapsto (W_{\tilde{\sigma}_0} f)(g)$ ,  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))' \rightarrow \mathbb{C}$ , is a well defined linear mapping. To prove that it is continuous let  $B$  be a bounded subset of  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ . Thus for each  $a > 1$  there exists  $C > 0$  such that  $|g_k| \leq C a^{|k|^{1/(2\alpha)}}$ ,  $\forall k \in \mathbb{N}_0^d$ ,  $\forall g \in B$ . Hence,

$$\sup_{g \in B} |(W_{\tilde{\sigma}_0} f)(g)| < \infty,$$

i.e.  $W_{\tilde{\sigma}_0} f$  maps bounded subsets in  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$  into bounded subsets of  $\mathbb{C}$ . Since  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$  is bornological,  $g \mapsto (W_{\tilde{\sigma}_0} f)(g)$  is continuous, hence  $W_{\tilde{\sigma}_0} f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  ( $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  is reflexive). Now we conclude that  $W_{\tilde{\sigma}_0} f = \sum_{n \in \mathbb{N}_0^d} f_n \sigma_n h_n$  (this is exactly Hermite expansion of  $W_{\tilde{\sigma}_0} f$ ;  $\{f_n \sigma_n\}_n \in \mathfrak{s}^{2\alpha}$ ). Thus, the mapping  $f \mapsto W_{\tilde{\sigma}_0} f$ ,  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))' \rightarrow \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ , is well defined and linear. Arguing similarly as before, one can prove that when  $f$  varies in a bounded subset  $B$  of  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ , the set  $\{\{f_k \sigma_k\}_{k \in \mathbb{N}_0^d} | f \in B\}$  is bounded in  $\mathfrak{s}^{2\alpha}$ . Thus  $\{W_{\tilde{\sigma}_0} f | f \in B\}$  is bounded in  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ . As  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$  is bornological, the mapping  $f \mapsto W_{\tilde{\sigma}_0} f$ ,  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))' \rightarrow \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ , is continuous. Observe that  $W_{\tilde{\sigma}_0} f$  coincides with the Weyl transform of  $f$  when  $f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  (cf. (30)).

If  $\sigma_j \rightarrow \sigma$  as  $j \rightarrow \infty$ , in  $G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d)$ , Theorem 6.1 implies that  $\{\sigma_{n,j}\}_{n \in \mathbb{N}_0^d} \rightarrow \{\sigma_n\}_{n \in \mathbb{N}_0^d}$  as  $j \rightarrow \infty$  in  $\mathfrak{s}^{2\alpha}$  and since the latter is a  $(DFN)$ -space, the convergence also holds in  $\mathfrak{s}^{2\alpha, a}$  for some  $a > 1$ . Thus, for each fixed  $f \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ ,  $\{f_n \sigma_{n,j}\}_n \rightarrow \{f_n \sigma_n\}_n$  in  $\mathfrak{s}^{2\alpha}$ . Hence,  $\sum_{n \in \mathbb{N}_0^d} f_n \sigma_{n,j} h_n \rightarrow \sum_{n \in \mathbb{N}_0^d} f_n \sigma_n h_n$  in  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ . We obtain  $W_{\tilde{\sigma}_0, j} \rightarrow W_{\tilde{\sigma}_0}$  in the topology of simple convergence in  $\mathcal{L}((\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))', \mathcal{S}_\alpha^\alpha(\mathbb{R}^d))$ . Now, the Banach-Steinhaus theorem implies that the convergence holds in the topology of precompact convergence. Since  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$  is a Montel space, the convergence also holds in the strong topology of  $\mathcal{L}((\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))', \mathcal{S}_\alpha^\alpha(\mathbb{R}^d))$ .  $\square$   $\square$

By similar arguments, one can prove the following theorem.

**Theorem 7.4.** *Let  $\sigma(\rho) \in \mathcal{S}(\mathbb{R}_+^d)$  and denote by  $\sigma_0(\rho) = \sigma(2\rho)$ ,  $\rho \in \mathbb{R}_+^d$ . Let  $\tilde{\sigma}, \tilde{\sigma}_0 \in \mathcal{S}(\mathbb{R}^{2d})$  be the functions defined in (28). Then the Weyl pseudo-differential operator  $W_{\tilde{\sigma}_0}$  extends to a continuous mapping  $W_{\tilde{\sigma}_0} : (\mathcal{S}(\mathbb{R}^d))' \rightarrow \mathcal{S}(\mathbb{R}^d)$ . If  $f, g \in (\mathcal{S}(\mathbb{R}^d))'$  and*

$$f_k = \langle f, h_k \rangle, g_k = \langle g, h_k \rangle \text{ and } \sigma_k = (2\pi)^{d/2} (-1)^{|k|} 2^{-d} \int_{\mathbb{R}_+^d} \sigma(\rho) \mathcal{L}_k(\rho) d\rho,$$

*then  $(W_{\tilde{\sigma}_0} f)(g) = (2\pi)^{-d/2} \sum_{k \in \mathbb{N}_0^d} f_k g_k \sigma_k$ . Moreover, if  $\sigma_{0,j}(\eta) \rightarrow \sigma_0(\eta)$  in  $\mathcal{S}(\mathbb{R}_+^d)$  as  $j \rightarrow \infty$  then  $W_{\tilde{\sigma}_{0,j}} \rightarrow W_{\tilde{\sigma}_0}$  in the strong topology of  $\mathcal{L}((\mathcal{S}(\mathbb{R}^d))', \mathcal{S}(\mathbb{R}^d))$ .*

Let  $\alpha \geq 1/2$ . If  $\sigma$  is a measurable function on  $\mathbb{R}_+^d$  such that  $\sigma(\rho)/(1+\rho)^{n/2} \in L^2(\mathbb{R}_+^d)$  for some  $n \in \mathbb{N}_0^d$  then one easily verifies that  $\sigma \in (\mathcal{S}(\mathbb{R}_+^d))'$ . Since the canonical inclusion  $G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d) \rightarrow \mathcal{S}(\mathbb{R}_+^d)$  is continuous and dense,  $(\mathcal{S}(\mathbb{R}_+^d))'$  is continuously injected into  $(G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$ , hence  $\sigma \in (G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$ .

**Lemma 7.5.** *Let  $\alpha \geq 1/2$  and  $\sigma_n$ ,  $n \in \mathbb{N}_0^d$ , be measurable functions on  $\mathbb{R}_+^d$  such that  $\sigma_n(\rho)/(1+\rho)^{n/2} \in L^2(\mathbb{R}_+^d)$ , for all  $n \in \mathbb{N}_0^d$  and for each  $A > 0$ ,*

$$\sum_{n \in \mathbb{N}_0^d} \left\| \sigma_n(\rho)/(1+\rho)^{n/2} \right\|_{L^2(\mathbb{R}_+^d)} A^{|n|} n^{\alpha n} < \infty.$$

*Then  $\sum_{n \in \mathbb{N}_0^d} \sigma_n$  converges absolutely in  $(G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$ .*

*For each  $n \in \mathbb{N}_0^d$ ,  $\tilde{\sigma}_n(x, \xi) = \sigma_n(2v(x, \xi))$  is measurable on  $\mathbb{R}^{2d}$  and  $\tilde{\sigma}_n(x, \xi)/(1+2v(x, \xi))^{n/2} \in L^2(\mathbb{R}^{2d})$ . Moreover,  $\sum_{n \in \mathbb{N}_0^d} \tilde{\sigma}_n(x, \xi)$  converges absolutely in  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d}))'$ .*

*Proof.* To prove that  $\sum_{n \in \mathbb{N}_0^d} \sigma_n$  converges absolutely in  $(G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$  let  $B$  be bounded subset of  $G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d)$ . For each  $f \in B$  denote by  $a_{n,f} = \langle f, \mathcal{L}_n \rangle$ . By Theorem 6.1,  $\{\{a_{n,f}\}_{n \in \mathbb{N}_0^d} | f \in B\}$  is bounded in  $\mathfrak{s}^{2\alpha}$  and hence also bounded in  $\mathfrak{s}^{2\alpha, a}$  for some  $a > 1$ , i.e. there exists  $C_0 > 0$  such that  $|a_{n,f}| \leq C_0 a^{-|n|^{1/(2\alpha)}}$  for all  $f \in B$ . For  $f \in B$ ,  $n \in \mathbb{N}_0^d$ , we have

$$\begin{aligned} |\langle \sigma_n, f \rangle| &\leq \sum_{k \in \mathbb{N}_0^d} |a_{k,f}| \int_{\mathbb{R}_+^d} |\sigma_n(\rho)| |\mathcal{L}_k(\rho)| d\rho \\ &\leq C_0 \left\| \sigma_n(\rho)/(1+\rho)^{n/2} \right\|_{L^2(\mathbb{R}_+^d)} \sum_{k \in \mathbb{N}_0^d} a^{-|k|^{1/(2\alpha)}} \sum_{m \leq n} \binom{n}{m} \|\rho^{m/2} \mathcal{L}_k\|_{L^2(\mathbb{R}_+^d)}. \end{aligned}$$

As in the first part of the proof of Proposition 5.1, by (11), there exist  $C_1, A > 1$  which depend on  $a$  but not on  $n \in \mathbb{N}_0^d$  such that

$$\sum_{k \in \mathbb{N}_0^d} a^{-|k|^{1/(2\alpha)}} \sum_{m \leq n} \binom{n}{m} \|\rho^{m/2} \mathcal{L}_k\|_{L^2(\mathbb{R}_+^d)} \leq C_1 A^{|n|} n^{\alpha n}.$$

Hence, by the assumption on  $\sigma_n$ ,  $n \in \mathbb{N}_0^d$ , we have  $\sum_{n \in \mathbb{N}_0^d} \sup_{f \in B} |\langle \sigma_n, f \rangle| < \infty$ , i.e.  $\sum_{n \in \mathbb{N}_0^d} \sigma(\rho)$  converges absolutely in  $(G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$ .

Next we prove that for each  $n \in \mathbb{N}_0^d$ ,  $\tilde{\sigma}_n$  is measurable on  $\mathbb{R}^{2d}$ . Firstly, we show the following:



Let  $v_1 : \mathbb{R}^{2d} \rightarrow \overline{\mathbb{R}_+^d}$  be defined by  $v_1(x, \xi) = (2x_1^2 + 2\xi_1^2, \dots, 2x_d^2 + 2\xi_d^2)$ . If  $g : \overline{\mathbb{R}_+^d} \rightarrow \mathbb{C}$  is measurable then  $f : \mathbb{R}^{2d} \rightarrow \mathbb{C}$ ,  $f = g \circ v_1$ , is also measurable.

For brevity in notation we denote by  $\lambda_d$  and  $\lambda_{2d}$  the Lebesgue measure on  $\mathbb{R}^d$  and  $\mathbb{R}^{2d}$ , respectively. We prove that if  $N \subseteq \overline{\mathbb{R}_+^d}$  with  $\lambda_d(N) = 0$  then  $\lambda_{2d}(v_1^{-1}(N)) = 0$ . Observe that this implies the measurability of  $f$  since every measurable set is the union of a Borel set and a set of measure zero and the preimage of every Borel set under  $v_1$  is Borel set (since  $v_1$  is continuous). Let  $N \subseteq \overline{\mathbb{R}_+^d}$  with  $\lambda_d(N) = 0$ . Denote by  $N_1 = N \cap \mathbb{R}_+^d$  and  $N_2 = N \setminus N_1$ . Obviously  $\lambda_{2d}(v_1^{-1}(\overline{\mathbb{R}_+^d} \setminus \mathbb{R}_+^d)) = 0$ , thus  $v_1^{-1}(N_2)$  is measurable and has measure zero. It remains to prove that  $\lambda_{2d}(v_1^{-1}(N_1)) = 0$ . Let  $\varepsilon > 0$  be arbitrary but fixed. Since  $\lambda_d(N_1) = 0$ , there exists an open set  $O \subseteq \mathbb{R}_+^d$ , such that  $N_1 \subseteq O$  and  $\lambda_d(O) < \varepsilon/\pi^d$ . There exist countable number of cubes  $B(\rho^{(j)}, r_j) = \{\rho \in \mathbb{R}_+^d \mid \rho_l^{(j)} \leq \rho_l < \rho_l^{(j)} + r_j, l = 1, \dots, d\}$ ,  $j \in \mathbb{N}$ , which are pairwise disjoint and  $O = \bigcup_{j \in \mathbb{N}} B(\rho^{(j)}, r_j)$  (cf. [23, p. 50]). Observe that

$$\varepsilon/\pi^d > \lambda_d(O) = \sum_{j \in \mathbb{N}} \lambda_d(B(\rho^{(j)}, r_j)) = \sum_{j \in \mathbb{N}} r_j^d$$

and

$$v_1^{-1}(B(\rho^{(j)}, r_j)) = \prod_{l=1}^d \left\{ (x_l, \xi_l) \mid \rho_l^{(j)}/2 \leq x_l^2 + \xi_l^2 < \rho_l^{(j)}/2 + r_j/2 \right\}.$$

Thus  $\lambda_{2d}(v_1^{-1}(B(\rho^{(j)}, r_j))) = r_j^d \pi^d / 2^d$ . Hence  $\lambda_{2d}(v_1^{-1}(O)) = \sum_{j \in \mathbb{N}} r_j^d \pi^d / 2^d < \varepsilon$ . Since  $\varepsilon > 0$  is arbitrary, we conclude that  $v_1^{-1}(N_1)$  is measurable and it has measure zero.

This fact readily implies the measurability of  $\tilde{\sigma}_n$ . Moreover,

$$\left\| \tilde{\sigma}_n(x, \xi) / (\mathbf{1} + 2v(x, \xi))^{n/2} \right\|_{L^2(\mathbb{R}^{2d})}^2 = 2^{-d} \pi^d \left\| \sigma_n(\rho) / (\mathbf{1} + \rho)^{n/2} \right\|_{L^2(\mathbb{R}_+^d)}^2.$$

Clearly,  $\tilde{\sigma}_n \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d}))'$  for each  $n \in \mathbb{N}_0^d$ . To prove that  $\sum_{n \in \mathbb{N}_0^d} \tilde{\sigma}_n$  converges absolutely in  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d}))'$ , let  $B$  be a bounded subset of  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d})$ . As the latter space is the inductive limit of  $\varinjlim_{A \rightarrow \infty} \mathcal{S}_{\alpha, A}^{\alpha, A}(\mathbb{R}^{2d})$  with compact linking mappings, there exist  $C, A \geq 1$  such that

$$\left\| x^n \xi^m D_x^p D_\xi^q f(x, \xi) \right\|_{L^2(\mathbb{R}^{2d})} \leq C A^{|n+m+p+q|} n!^\alpha m!^\alpha p!^\alpha q!^\alpha,$$

$\forall n, m, p, q \in \mathbb{N}_0^d, \forall f \in B$ . For  $f \in B$ , we have

$$\begin{aligned} |\langle \tilde{\sigma}_n, f \rangle| &\leq \left\| \tilde{\sigma}_n(x, \xi) / (\mathbf{1} + 2v(x, \xi))^{n/2} \right\|_{L^2(\mathbb{R}^{2d})} \left\| f(x, \xi) (\mathbf{1} + 2v(x, \xi))^{n/2} \right\|_{L^2(\mathbb{R}^{2d})} \\ &\leq \pi^d 2^{|n|} \left\| \sigma_n(\rho) / (\mathbf{1} + \rho)^{n/2} \right\|_{L^2(\mathbb{R}_+^d)}^2 \sum_{m+k+p=n} \frac{n!}{m!k!p!} \left\| x^m \xi^k f(x, \xi) \right\|_{L^2(\mathbb{R}^{2d})} \\ &\leq C \pi^d (6A)^{|n|} n!^\alpha \left\| \sigma_n(\rho) / (\mathbf{1} + \rho)^{n/2} \right\|_{L^2(\mathbb{R}_+^d)}^2. \end{aligned}$$

Hence, by the assumption in the lemma,  $\sum_{n \in \mathbb{N}_0^d} \sup_{f \in B} |\langle \tilde{\sigma}_n, f \rangle| < \infty$ , i.e.  $\sum_{n \in \mathbb{N}_0^d} \tilde{\sigma}_n$  absolutely converges in  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d}))'$ .  $\square$

Let  $\sigma_n$  and  $\tilde{\sigma}_n$ ,  $n \in \mathbb{N}_0^d$ , be as in the previous lemma and  $\tilde{\sigma}(x, \xi) = \sum_{n \in \mathbb{N}_0^d} \tilde{\sigma}_n(x, \xi) \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ . The Weyl pseudo-differential operator  $W_{\tilde{\sigma}}$  is a continuous mapping from  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  into  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ . In this case, we obtain improvement with the following result.

**Theorem 7.6.** *Let  $\alpha \geq 1/2$ . Let  $\sigma_n(\rho)$  and  $\tilde{\sigma}_n(x, \xi) = \sigma_n(2v(x, \xi))$ ,  $n \in \mathbb{N}_0^d$ , be as in Lemma 7.5. Then the Weyl pseudo-differential operator  $W_{\tilde{\sigma}}$  with a symbol  $\tilde{\sigma}(x, \xi) = \sum_{n \in \mathbb{N}_0^d} \tilde{\sigma}_n(x, \xi) \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d}))'$  is a continuous mapping from  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  into  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  and it extends to a continuous mapping from  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$  to  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ .*

Assume that for each  $j \in \mathbb{N}$ ,  $\sigma_n^{(j)} \in (G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$ ,  $n \in \mathbb{N}_0^d$ , be as above and denote by  $\sigma^{(j)} = \sum_{n \in \mathbb{N}_0^d} \sigma_n^{(j)} \in (G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$ . If  $\sigma^{(j)} \rightarrow \sigma$  in  $(G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$  with  $\sigma$  as above, then  $W_{\tilde{\sigma}^{(j)}} \rightarrow W_{\tilde{\sigma}}$  in the strong topology of  $\mathcal{L}(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d), \mathcal{S}_\alpha^\alpha(\mathbb{R}^d))$  and  $\mathcal{L}((\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))', (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))')$ .

*Proof.* Denote  $\sigma = \sum_{n \in \mathbb{N}_0^d} \sigma_n \in (G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$  (cf. Lemma 7.5). Let  $f, g \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ . Denote  $f_k = \langle f, h_k \rangle$ ,  $g_k = \langle g, h_k \rangle$  and  $s_k = (2\pi)^{d/2}(-1)^{|k|}2^{-d}\langle \sigma, \mathcal{L}_k \rangle$ . Similarly as in the first part of the proof of Theorem 7.3, one obtains

$$(W_{\tilde{\sigma}}f)(g) = (2\pi)^{-d/2} \sum_{(m,k) \in \mathbb{N}_0^{2d}} f_m g_k \langle \tilde{\sigma}, \psi_{m,k} \rangle,$$

where  $\psi_{m,k} = W(h_m, h_k)$  and the sum converges absolutely. Next,

$$\langle \tilde{\sigma}(x, \xi), \psi_{m,k}(x, \xi) \rangle = \sum_{n \in \mathbb{N}_0^d} \int_{\mathbb{R}^{2d}} \sigma_n(2v(x, \xi)) \psi_{m,k}(x, \xi) dx d\xi.$$

By the same technique as in the proof of Theorem 7.3,

$$\int_{\mathbb{R}^{2d}} \sigma_n(2v(x, \xi)) \psi_{m,k}(x, \xi) dx d\xi = C_{n,m,k} \prod_{r=1}^d \int_{-\pi}^{\pi} e^{-i(m_r - k_r)\theta_r} d\theta_r.$$

Thus  $\langle \tilde{\sigma}, \psi_{m,k} \rangle = 0$  for  $m \neq k$ . Moreover,

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} \sigma_n(2v(x, \xi)) \psi_{k,k}(x, \xi) dx d\xi \\ &= (2\pi)^{d/2}(-1)^{|k|}2^d \int_{\mathbb{R}_+^d} \sigma_n(2\rho_1^2, \dots, 2\rho_d^2) L_k(2\rho_1^2, \dots, 2\rho_d^2) e^{-|\rho|^2} \rho^1 d\rho \\ &= (2\pi)^{d/2}(-1)^{|k|}2^{-d} \langle \sigma_n, \mathcal{L}_k \rangle. \end{aligned}$$

Thus,  $\langle \tilde{\sigma}(x, \xi), \psi_{k,k}(x, \xi) \rangle = (2\pi)^{d/2}(-1)^{|k|}2^{-d} \langle \sigma, \mathcal{L}_k \rangle = s_k$ . Hence, we obtain

$$(W_{\tilde{\sigma}}f)(g) = (2\pi)^{-d/2} \sum_{k \in \mathbb{N}_0^d} f_k g_k s_k$$

and the series converges absolutely since  $\{f_k\}_{k \in \mathbb{N}_0^d}, \{g_k\}_{k \in \mathbb{N}_0^d} \in \mathfrak{s}^{2\alpha}$  (see Proposition 2.3) and  $\{s_k\}_{k \in \mathbb{N}_0^d} \in (\mathfrak{s}^{2\alpha})'$  (see Theorem 6.2). Observe that for each  $n \in \mathbb{N}_0^d$ ,  $(W_{\tilde{\sigma}}f)(h_n) = f_n s_n$ . Since  $\{s_n\}_{n \in \mathbb{N}_0^d} \in (\mathfrak{s}^{2\alpha})'$  and  $\{f_n\}_{n \in \mathbb{N}_0^d} \in \mathfrak{s}^{2\alpha}$ , we have  $\{f_n s_n\}_{n \in \mathbb{N}_0^d} \in \mathfrak{s}^{2\alpha}$ , i.e.  $W_{\tilde{\sigma}}f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  (by Proposition 2.3). We conclude that  $f \mapsto W_{\tilde{\sigma}}f$ ,  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \rightarrow \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ , is a well defined linear mapping. Moreover,  $W_{\tilde{\sigma}}f =$

$\sum_{n \in \mathbb{N}_0^d} f_n s_n h_n$ . To prove the continuity let  $B$  be a bounded subset of  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ . As  $\{s_k\}_{k \in \mathbb{N}_0^d} \in (\mathfrak{s}^{2\alpha})'$ , the set  $\{\{f_n s_n\}_{n \in \mathbb{N}_0^d} \mid f \in B\}$  is bounded in  $\mathfrak{s}^{2\alpha}$ , thus  $\{W_{\tilde{\sigma}} f \mid f \in B\}$  is bounded in  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ . As  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  is bornological,  $f \mapsto W_{\tilde{\sigma}} f$ ,  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \rightarrow \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ , is continuous. By similar technique, one proves that for each  $f \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ ,  $W_{\tilde{\sigma}} f = \sum_{n \in \mathbb{N}_0^d} f_n s_n h_n \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$  and the mapping,  $f \mapsto W_{\tilde{\sigma}} f$ ,  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))' \rightarrow (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ , is continuous.

Let  $\sigma, \sigma^{(j)} \in (G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$ ,  $j \in \mathbb{N}$ , be as assumed in the theorem, with  $\sigma^{(j)} \rightarrow \sigma$  in  $(G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$ . In order to prove  $W_{\tilde{\sigma}^{(j)}} \rightarrow W_{\tilde{\sigma}}$  in the strong topology of  $\mathcal{L}(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d), \mathcal{S}_\alpha^\alpha(\mathbb{R}^d))$  (resp. in the strong topology of  $\mathcal{L}((\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))', (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))')$ ) it is enough to prove that for each  $f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  (resp. for each  $f \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ ),  $W_{\tilde{\sigma}^{(j)}} f \rightarrow W_{\tilde{\sigma}} f$  in  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  (resp. in  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ ) since in this case the Banach-Steinhaus theorem implies convergence in the topology of precompact convergence and as  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  (resp.  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ ) is Montel the convergence also holds in the strong topology. Thus for the fixed  $f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  (resp.  $f \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ ). Theorem 6.2 implies that  $\{s_k^{(j)}\}_{k \in \mathbb{N}_0^d} \rightarrow \{s_k\}_{k \in \mathbb{N}_0^d}$  in  $(\mathfrak{s}^{2\alpha})'$ . But then  $\{f_k s_k^{(j)}\}_{k \in \mathbb{N}_0^d} \rightarrow \{f_k s_k\}_{k \in \mathbb{N}_0^d}$  in  $\mathfrak{s}^{2\alpha}$  (resp. in  $(\mathfrak{s}^{2\alpha})'$ ), i.e.  $W_{\tilde{\sigma}^{(j)}} f \rightarrow W_{\tilde{\sigma}} f$  in  $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$  (resp. in  $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ ).  $\square$

By the similar arguments, one can proof the following theorem.

**Theorem 7.7.** *Let  $\sigma$  be a measurable function on  $\mathbb{R}_+^d$  such that there exists  $n \in \mathbb{N}_0^d$  for which  $\sigma(\rho)/(\mathbf{1} + \rho)^n \in L^2(\mathbb{R}_+^d)$ . Then  $\tilde{\sigma}(x, \xi) = \sigma(2v(x, \xi)) \in (\mathcal{S}(\mathbb{R}^{2d}))'$ . The Weyl pseudo-differential operator  $W_{\tilde{\sigma}}$  is a continuous mapping from  $\mathcal{S}(\mathbb{R}^d)$  into  $\mathcal{S}(\mathbb{R}^d)$  and it extends to continuous mapping from  $(\mathcal{S}(\mathbb{R}^d))'$  into  $(\mathcal{S}(\mathbb{R}^d))'$ .*

*Let  $\sigma^{(j)}$ ,  $j \in \mathbb{N}$ , be measurable functions on  $\mathbb{R}_+^d$  such that for each  $j \in \mathbb{N}$  there exists  $n^{(j)} \in \mathbb{N}_0^d$  for which  $\sigma_j(\rho)/(\mathbf{1} + \rho)^{n^{(j)}} \in L^2(\mathbb{R}_+^d)$ . If  $\sigma$  is a measurable function on  $\mathbb{R}_+^d$  with the properties stated above and if  $\sigma^{(j)} \rightarrow \sigma$  in  $(\mathcal{S}(\mathbb{R}_+^d))'$ , then  $W_{\tilde{\sigma}^{(j)}} \rightarrow W_{\tilde{\sigma}}$  in the strong topology of  $\mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$  and  $\mathcal{L}((\mathcal{S}(\mathbb{R}^d))', (\mathcal{S}(\mathbb{R}^d))')$ .*

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